# STRONG CONVERGENCE THEOREMS FOR FAMILIES OF NONLINEAR MAPPINGS WITH GENERALIZED PARAMETERS IN HILBERT SPACES

By

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## Abstract

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**Project name** : Strong convergence theorems for families of nonlinear mappings with generalized parameters in Hilbert spaces

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In this research, motivated and inspired by the above facts, we introduce a new iterative scheme for finding a common element of the set of fixed points of three nonexpansive mappings, and the set of solutions of a mixed equilibrium problem in a real Hilbert space. Strong convergence results are derived under suitable conditions in a real Hilbert space. We introduce a new iterative scheme for finding common solutions of a variational inequality for an inverse-strongly accretive mapping and the solutions of a fixed point problem for a nonexpansive semigroup by using the modified Mann iterative method. We shall prove the strong convergence theorem in a $q$-uniformly smooth Banach spaces under some parameters controlling conditions.

**Keywords :** Fixed point; Hilbert space.

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CHAPTER I

# INTRODUCTION

## 1.1 Background of research

The fixed-point iteration process for nonlinear operators in Hilbert spaces and Banach spaces including Mann, Halpern and Isikawa iterations process have been studied extensively by many authors to approximate fixed point of various classes of operators in both Hilbert spaces and Banach spaces. In 1952, Mann [42] defined Mann iteration in a matrix formulation. In 1967, Halpern [17] introduced the new innovation iteration process which resemble in Mann's iteration.

In 1994, equilibrium problems were introduced by Blum and Oettli [5] and by Noor and Oettli [47] as optimization problems and generalizations of variational inequalities. The equilibrium problem theory provides a novel and united treatment of a wide class of problems which arise in finance, economics, ecology, elasticity, transportation, network and optimization. This theory has had a great impact and influence in the development of several branches of pure and applied sciences.

## 1.2 Objective

The objectives of research project

1.2.1 We construct an iterative scheme of nonexpansive mappings in Hilbert spaces and we prove strong convergence theorems.

1.2.2 Apply iterative scheme to prove strong convergence theorems and reduce condition to other theorems.

## 1.3 Benefits of research

The benefits of research project

1.3.1. To gain the method for construct an iterative scheme of nonexpansive mappings in Hilbert spaces.

1.3.2. To apply iterative scheme to prove strong convergence theorems and reduce condition to other theorems.

# CHAPTER II

# PRELIMINARIES

In this chapter, we give some definitions, notations, and some useful results that will be used in the later chapters.

## 2.1 Basic results.

**Definition 2.1.** Let $X$ be a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : X \longrightarrow \mathbb{R}$ is said to be *a norm on* $X$ if it satisfies the following conditions:

(1) $\|x\| \geq 0, \forall x \in X$;

(2) $\|x\| = 0 \Leftrightarrow x = 0$;

(3) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$;

(4) $\|\alpha x\| = |\alpha|\|x\|, \forall x \in X$ and $\forall \alpha \in \mathbb{K}$.

**Definition 2.2.** Let $(X, \|\cdot\|)$ be a normed space.

(1) A sequence $\{x_n\} \subset X$ is said to *converge strongly* in $X$ if there exists $x \in X$ such that $\lim_{n \longrightarrow \infty} \|x_n - x\| = 0$. That is, if for any $\epsilon > 0$ there exists a positive integer $N$ such that $\|x_n - x\| < \epsilon, \forall n \geq N$. We often write $\lim_{n \longrightarrow \infty} x_n = x$ or $x_n \longrightarrow x$ to mean that $x$ is the limit of the sequence $\{x_n\}$.

(2) A sequence $\{x_n\} \subset X$ is said to be *a Cauchy sequence* if for any $\epsilon > 0$ there exists a positive integer $N$ such that $\|x_m - x_n\| < \epsilon, \forall\, m, n \geq N$. That is, $\{x_n\}$ is *a Cauchy sequence* in $X$ if and only if $\|x_m - x_n\| \longrightarrow 0$ as $m, n \longrightarrow \infty$.

**Definition 2.3.** A normed space $X$ is called *complete* if every Cauchy sequence in $X$ converges to an element in $X$.

**Definition 2.4.** An element $x \in C$ is said to be a *fixed point* of a mapping $S : C \longrightarrow C$ proved $Sx = x$. The set of all fixed point of $S$ is denoted by $F(S) = \{x \in$

$C : Sx = x\}$.

**Definition 2.5.** Let $F$ and $X$ be linear spaces over the field $\mathbb{K}$.

(1) A mapping $T : F \longrightarrow X$ is called *a linear operator* if $T(x+y) = Tx+Ty$ and $T(\alpha x) = \alpha Tx, \forall x, y \in F$, and $\forall \alpha \in \mathbb{K}$.

(2) A mapping $T : F \longrightarrow \mathbb{K}$ is called *a linear functional on F* if $T$ is a linear operator.

**Definition 2.6.** Let $F$ and $X$ be normed spaces over the field $\mathbb{K}$ and $T : X \longrightarrow F$ a linear operator. $T$ is said to be *bounded* on $X$, if there exists a real number $M > 0$ such that $\|T(x)\| \leq M\|x\|, \forall x \in X$.

**Definition 2.7.** Sequence $\{x_n\}_{n=1}^{\infty}$ in a normed linear space $X$ is said to be *a bounded sequence* if there exists $M > 0$; such that $\|x_n\| \leq M, \forall n \in \mathbb{N}$.

**Definition 2.8.** Let $F$ and $X$ be normed spaces over the field $\mathbb{K}$, $T : F \longrightarrow X$ an operator and $c \in F$. We say that $T$ is *continuous at c* if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|T(x) - T(c)\| < \epsilon$ whenever $\|x - c\| < \delta$ and $x \in F$. If $T$ is continuous at each $x \in F$, then $T$ is said to be *continuous on F*.

**Definition 2.9.** A subset $C$ of a normed linear space $X$ is said to be *convex subset in X* if $\lambda x + (1 - \lambda)y \in C$ for each $x, y \in C$ and for each scalar $\lambda \in [0, 1]$.

## 2.2 Inner product spaces

**Definition 2.10.** The real-valued function of two variables $\langle \cdot, \cdot \rangle : X \times X \longrightarrow \mathbb{R}$ is called *inner product* on a real vector space $X$ if it satisfies the following conditions:

1) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y, z \in X$ and all real number $\alpha$ and $\beta$;

2) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in X$; and

3) $\langle x, x \rangle \geq 0$ for each $x \in X$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

A *real inner product space* is a real vector space equipped with an inner product.

*Remark* 2.11. Every inner product space is a normed space with respect to the norm $\|x\| = |\langle x, x\rangle|^{\frac{1}{2}}, x, y \in X$.

**Definition 2.12.** A *Hilbert space* is an inner product space which is complete under the norm induced by its inner product.

**Definition 2.13.** A sequence $\{x_n\}$ in a Hilbert space $H$ is said to *converge weakly* to a point $x$ in $H$ if $\lim_{n\longrightarrow\infty}\langle x_n, y\rangle = \langle x, y\rangle$ for all $y \in H$. The notation $x_n \rightharpoonup x$ is sometimes used to denote this kind of convergence.

**Definition 2.14.** The *metric (nearest point) projection* $P_C$ from a Hilbert space $H$ to a closed convex subset $C$ of $H$ is defined as follows: Given $x \in H$, $P_C x$ is the only point in $C$ with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

**Lemma 2.15.** *Let $H$ be a real Hilbert space, $C$ a closed convex subset of $H$. Given $x \in H$ and $z \in C$. Then*

*1) $z = P_C x$ if and only if there holds the inequality*

$$\langle x - z, z - y\rangle \geq 0, \forall y \in C.$$

*2) $\|P_C x - P_C y\| \leq \|x - y\|,\ \forall x, y \in H$.*

*3) $\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$*

## 2.3 Hilbert spaces

**Definition 2.16.** The real-value function of two variables $\langle\cdot,\cdot\rangle : X \times X \longrightarrow \mathbb{R}$ is called *inner product* on a real vector space $X$ if it satisfies the following conditions:

(1) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$ for all $x, y, z \in X$ and all real number $\alpha$ and $\beta$;

(2) $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in X$; and

(3) $\langle x, x\rangle \geq 0$ for each $x \in X$ and $\langle x, x\rangle = 0$ if and only if $x = 0$. A *real inner product space* is a real vector space equipped with an inner product.

**Definition 2.17.** A Hilbert spaces is an inner product space which is complete under the norm induced by its inner product.

**Definition 2.18.** The metric (nearest point) projection $P_C$ from a Hilbert space $H$ to a closed convex subset $C$ of $H$ is defined as follows: Given $x \in H$, $P_C x$ is the only point in $C$ with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

**Definition 2.19.** For every point $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

$P_C$ is called the metric projection of $H$ onto $C$. It is well known that $P_C$ is a firmly nonexpansive mapping of $H$ onto $C$ and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H \tag{2.1}$$

**Definition 2.20.** Let $X$ be a normed space, $\{x_n\} \subset X$ and $f : X \longrightarrow (-\infty, \infty]$. Then $f$ is said to be

1) *lower semicontinuous* on $X$ if for any $x_0 \in X$, $f(x_0) \leqslant \liminf_{n\longrightarrow\infty} f(x_n)$ whenever $x_n \longrightarrow x_0$.

2) *upper semi (or hemi) continuous* on $X$ if for any $x_0 \in X$, $\limsup_{n\longrightarrow\infty} f(x_n) \leqslant f(x_0)$ whenever $x_n \longrightarrow x_0$.

3) *weakly lower semicontinuous* on $X$ if for any $x_0 \in X$, $f(x_0) \leqslant \liminf_{n\longrightarrow\infty} f(x_n)$ whenever $x_n \rightharpoonup x_0$.

4) *weakly upper semicontinuous* on $X$ if for any $x_0 \in X$, $\limsup_{n\longrightarrow\infty} f(x_n) \leqslant f(x_0)$ whenever $x_n \rightharpoonup x_0$.

**Definition 2.21.** Let $X$ be a normed space. A mapping $T : X \longrightarrow X$ is said to be *Lipschitzian* if there exists a constant $k \geq 0$ such that for all $x, y \in X$,

$$\|Tx - Ty\| \leq k\|x - y\|. \tag{2.2}$$

The smallest number $k$ for which (2.4) holds is called the *Lipschitz constant* of $T$ and $T$ is called *a contraction (nonexpansive mapping)* if $k \in (0, 1)$ (k=1).

**Definition 2.22.** An element $x \in X$ is said to be

1) a *fixed point* of a mapping $T : X \longrightarrow X$ provided $Tx = x$.

2) a *common fixed point* of two mappings $S, T : X \longrightarrow X$ provided $Sx = x = Tx$. The set of all fixed points of $T$ is denoted by $F(T)$.

**Lemma 2.23.** *Let $H$ be a real Hilbert space, $C$ a closed convex subset of $H$. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \forall z \in C.$$

**Definition 2.24.** *A Linear space or vector space* $X$ over the field $\mathbb{K}$ (The real field $\mathbb{R}$ or the complex field $\mathbb{C}$) is a set $X$ together with an internal binary operation "+" called *addition* and a *scalar multiplication* carrying $(\alpha, x)$ in $\mathbb{K} \times X$ to $\alpha x$ in $X$ satisfying the following for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{K}$:

1) $x + y = y + x$,

2) $(x + y) + z = x + (y + z)$,

3) there exists an element $0 \in X$ called the *zero vector* of $X$ such that $x + 0 = x$ for all $x \in X$,

4) for every element $x \in X$, there exists an element $-x \in X$ called the *additive inverse or the negative* of $x$ such $x + (-x) = 0$,

5) $\alpha(x + y) = \alpha x + \alpha y$,

6) $(\alpha + \beta)x = \alpha x + \beta x$,

7) $(\alpha\beta)x = \alpha(\beta x)$,

8) $1 \cdot x = x$.

The elements of a vector space $X$ are called *vector*, and the elements of $\mathbb{K}$ called *scalars*. In the sequel, unless otherwise stated, $X$ denotes a linear space over field $\mathbb{R}$.

**Definition 2.25.** A subset $C$ of a linear space $X$ is said to be *a convex set in* $X$ if $\lambda x + (1-\lambda)y \in C$ for each $x, y \in C$ and for each scalar $\lambda \in [0,1]$.

**Definition 2.26.** Let $X$ and $Y$ be normed spaces over the field $\mathbb{K}$, $T : X \longrightarrow Y$ an operator and $x_0 \in X$. We say that $T$ is *continuous at* $x_0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|T(x) - T(x_0)\| < \varepsilon$ whenever $\|x - x_0\| < \delta$ and $x \in X$. If $T$ is continuous at each $x \in X$, then $T$ is said to be *continuous on* $X$.

*Remark* 2.27. A linear operator on a normed space is bounded if and only if it is continuous. We now denote the set of all continuous (or bounded) linear operators of $X$ into $Y$ by $L(X, Y)$.

**Definition 2.28.** Let $X$ be a normed space. A mapping $T : X \longrightarrow X$ is said to be *Lipschitzian* if there exists a constant $k \geq 0$ such that for all $x, y \in X$,

$$\|Tx - Ty\| \leq k\|x - y\|. \tag{2.3}$$

The smallest number $k$ for which (2.4) holds is called the *Lipschitz constant* of $T$ and $T$ is called *a contraction (nonexpansive mapping)* if $k \in (0, 1)$ (k=1).

## 2.4 Normed spaces and Banach spaces

**Definition 2.29.** [10] Let $X$ be a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : X \longrightarrow \mathbb{K}$ is said to be *a norm on* $X$ if it satisfies the following conditions:

1) $\|x\| \geq 0, \forall x \in X;$

2) $\|x\| = 0 \Leftrightarrow x = 0;$

3) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in E;$

4) $\|\alpha x\| = |\alpha|\|x\|, \forall x \in E$ and $\forall \alpha \in \mathbb{K}$.

We use the notation $\|\cdot\|$ for norm.

**Definition 2.30.** [10] Let $(X, \|\cdot\|)$ be a normed space.

1) A sequence $\{x_n\} \subset X$ is said to *converge strongly* in $X$ if there exists $x \in X$ such that $\lim_{n\longrightarrow\infty} \|x_n - x\| = 0$. That is, if for any $\varepsilon > 0$ there exists a positive integer $N$ such that $\|x_n - x\| < \varepsilon, \forall n \geq N$. We often write $\lim_{n\longrightarrow\infty} x_n = x$ or $x_n \longrightarrow x$ to mean that $x$ is the limit of the sequence $\{x_n\}$.

2) A sequence $\{x_n\} \subset X$ is said to be *a Cauchy sequence* if for any $\varepsilon > 0$ there exists a positive integer $N$ such that $\|x_m - x_n\| < \varepsilon, \forall\, m, n \geq N$. That is, $\{x_n\}$ is *a Cauchy sequence* in $X$ if and only if $\|x_m - x_n\| \longrightarrow 0$ as $m, n \longrightarrow \infty$.

3) A sequence $\{x_n\} \subset X$ is said to be *a bounded sequence* if there exists $M > 0$ such that $\|x_n\| \leqslant M, \forall n \in \mathbb{N}$.

**Definition 2.31.** [10] A normed space $X$ is called to be *complete* if every Cauchy sequence in $X$ converges to an element in $X$.

**Definition 2.32.** [10] A complete normed linear space over field $\mathbb{K}$ is called *a Banach space over* $\mathbb{K}$

**Definition 2.33.** [10] Let $X$ and $Y$ be linear spaces over the field $\mathbb{K}$.

1) A mapping $T : X \longrightarrow Y$ is called *a linear operator* if $T(x + y) = Tx + Ty$ and $T(\alpha x) = \alpha Tx, \forall x, y \in X$, and $\forall \alpha \in \mathbb{K}$.

2) A mapping $T : X \longrightarrow \mathbb{K}$ is called *a linear functional on* $X$ if $T$ is a linear operator.

**Definition 2.34.** [10] Let $X$ and $Y$ be normed spaces over the field $\mathbb{K}$ and $T : X \longrightarrow Y$ a linear operator. $T$ is said to be *bounded* on $X$, if there exists a real number $M > 0$ such that $\|T(x)\| \leq M\|x\|, \forall x \in X$.

**Definition 2.35.** [10] Let $X$ and $Y$ be normed spaces over the field $\mathbb{K}$, $T : X \longrightarrow Y$ an operator and $x_0 \in X$. We say that $T$ is *continuous at* $x_0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|T(x) - T(x_0)\| < \varepsilon$ whenever $\|x - x_0\| < \delta$ and $x \in X$. If $T$ is continuous at each $x \in X$, then $T$ is said to be *continuous on* $X$.

**Definition 2.36.** [11] Let $X$ be a normed space, $\{x_n\} \subset X$ and $f : X \longrightarrow (-\infty, \infty]$. Then $f$ is said to be

1) *lower semicontinuous* on $X$ if for any $x_0 \in X$, $f(x_0) \leqslant \liminf_{n\longrightarrow\infty} f(x_n)$ whenever $x_n \longrightarrow x_0$.

2) *upper semi (or hemi) continuous* on $X$ if for any $x_0 \in X$, $\limsup_{n\longrightarrow\infty} f(x_n) \leqslant f(x_0)$ whenever $x_n \longrightarrow x_0$.

3) *weakly lower semicontinuous* on $X$ if for any $x_0 \in X$, $f(x_0) \leqslant \liminf_{n\longrightarrow\infty} f(x_n)$ whenever $x_n \rightharpoonup x_0$.

4) *weakly upper semicontinuous* on $X$ if for any $x_0 \in X$, $\limsup_{n\longrightarrow\infty} f(x_n) \leqslant f(x_0)$ whenever $x_n \rightharpoonup x_0$.

**Definition 2.37.** [10] Let $X$ be a normed space. A mapping $T : X \longrightarrow X$ is said to be *Lipschitzian* if there exists a constant $k \geq 0$ such that for all $x, y \in X$,

$$\|Tx - Ty\| \leq k\|x - y\|. \tag{2.4}$$

The smallest number $k$ for which (2.4) holds is called the *Lipschitz constant* of $T$ and $T$ is called *a contraction (nonexpansive mapping)* if $k \in (0, 1)$ (k=1).

**Definition 2.38.** [10] An element $x \in X$ is said to be

1) a *fixed point* of a mapping $T : X \longrightarrow X$ provided $Tx = x$.

2) a *common fixed point* of two mappings $S, T : X \longrightarrow X$ provided $Sx = x = Tx$. The set of all fixed points of $T$ is denoted by $F(T)$.

**Theorem 2.39.** (Banach contraction principle, [10]) *Every contraction mapping $T$ defined on a Banach space $X$ into itself has a unique fixed point $x^* \in X$.*

**Definition 2.40.** [10] Let $X$ be a normed space. Then the set of all bounded linear functionals on $X$ is called *a dual space* of $X$ and is denoted by $X^*$.

**Definition 2.41.** [10] A normed space $X$ is said to be *reflexive* if the *canonical mapping* $G : X \longrightarrow X^{**}$ (i.e. $G(x) = g_x$ for all $x \in X$ where $g_x(f) = f(x)$ for all $f \in X^*$) is surjective.

**Definition 2.42.** [11] A Banach space $X$ is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$

**Definition 2.43.** [12] A Banach space $X$ is said to be *uniformly convex* if for each $0 < \varepsilon \leqslant 2$, there is $\delta > 0$ such that $\forall x, y \in X$, the condition $\|x\| = \|y\| = 1$, and $\|x - y\| \geqslant \varepsilon$ imply $\|\frac{x+y}{2}\| \leqslant 1 - \delta$.

**Theorem 2.44.** [12] *Let $X$ be a Banach space. Then $X$ is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.*

**Definition 2.45.** [11] Let $X$ be a Banach space and $S = \{x \in X : \|x\| = 1\}$. Then $X$ is said to be *smooth* if the limit

$$\lim_{t \longrightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.5}$$

exists for all $x, y \in S$. It is also said to be *uniformly smooth* if the limit (2.5) is attained uniformly for $x, y \in S$.

*Remark* 2.46. [11] 1) $X$ is uniformly convex if and only if $X^*$ is uniformly smooth.

2) $X$ is smooth if and only if $X^*$ is strictly convex.

**Definition 2.47.** [11] Let $X^*$ be dual space of a Banach space $X$. The mapping $J : X \longrightarrow X^*$ defined by

$$J(x) = \{x^* \in X : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}, \text{for all } x \in X,$$

is called the *duality mapping of* $X$.

**Lemma 2.48.** [11] *Let $X$ be a strictly convex, smooth, and reflexive Banach space, and let $J$ be the duality mapping from $X$ into $X^*$. Then $J^{-1}$ is also single-valued, one-to-one, and surjective, and it is the duality mapping from $X^*$ into $X$.*

**Lemma 2.49.** [13] *Let $X$ be a reflexive Banach space and $X^*$ be strictly convex.*

*(i) The duality mapping $J : X \longrightarrow X^*$ is single-valued, surjective and bounded.*

*(ii) If $X$ and $X^*$ are locally uniformly convex, then $J$ is a homeomorphism, that is, $J$ and $J^{-1}$ are continuous single-valued mappings.*

**Definition 2.50.** [14] Let $p$ be a fixed real number with $p \geq 1$. A Banach space $X$ is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in (0, 2]$.

**Definition 2.51.** [11] For each $p > 1$, the *generalized duality mapping* $J_p : X \longrightarrow 2^{X^*}$ is defined by

$$J_p(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\} \tag{2.6}$$

for all $x \in X$.

*Remark* 2.52. [11] 1) $J = J_2$ is called the normalized duality mapping. If $X$ is a Hilbert space (the next section), then $J = I$, where $I$ is the identity mapping.

2) If $X$ is uniformly smooth, then $J$ is uniformly norm-to-norm continuous on each bounded subset of $E$.

**Definition 2.53.** [11] Let $S(E) = \{x \in E : \|x\| = 1\}$ denote the unit sphere of a Banach space $E$. A Banach space $E$ is said to have

- *a Gâteaux differentiable norm* (we also say that E is smooth), if the limit

$$\lim_{t \longrightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.7}$$

  exists for each $x, y \in S(E)$;

- *a uniformly Gâteaux differentiable norm* , if for each y in $S(E)$, the limit (2.7) is uniformly attained for $x \in S(E)$;

- *a Fréchet differentiable norm*, if for each $x \in S(E)$, the limit (2.7) is attained uniformly for $y \in S(E)$;

- *a uniformly Fréchet differentiable norm* (we also say that $E$ is uniformly smooth), if the limit (2.7) is attained uniformly for $(x, y) \in S(E) \times S(E)$.

**Definition 2.54.** [11] A Banach space $E$ is said to have Kadec-Klee property if a sequence $\{x_n\}$ of $E$ satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \longrightarrow \|x\|$, then $x_n \longrightarrow x$.

It is known that if $E$ is uniformly convex, then $E$ has the Kadec-Klee property.

**Definition 2.55.** [15] Let $X$ be a smooth Banach space. The function $\phi : X \times X \longrightarrow \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{2.8}$$

for all $x, y \in X$.

*Remark* 2.56. (1) $(\|y\| - \|x\|)^2 \leqslant \phi(y, x) \leqslant (\|y\| + \|x\|)^2$, for all $x, y \in X$.

(2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$, for all $x, y, z \in X$.

(3) $\phi(x, y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leqslant \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$, for all $x, y \in X$.

(4) In a Hilbert space $H$, we have $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$.

**Definition 2.57.** [16] Let $C$ be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space $X$, for any $x \in X$, there exists a point $x_0 \in C$ such that $\phi(x_0, x) = \min_{y \in C} \phi(y, x)$. The mapping $\Pi_C : X \longrightarrow C$ defined by $\Pi_C x = x_0$ is called the *generalized projection.*

The following are well-known results.

**Lemma 2.58.** [17] *Let $E$ be a reflexive, strictly convex and smooth Banach space, let $C$ be a nonempty closed convex subset of $E$ and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leqslant \phi(y, x)$$

*for all $y \in C$.*

**Lemma 2.59.** [16] *Let $C$ be a nonempty closed convex subset of a smooth Banach space $X$, let $x \in X$, and let $x_0 \in C$. Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0\rangle \geqslant 0$ for all $y \in C$.*

**Lemma 2.60.** [4] Let $C$ be a nonempty, closed and convex subset of a Hilbert space $H$ and let $T : C \longrightarrow C$ be a firmly nonexpansive mapping with $F(T) \neq \emptyset$. Then $\langle x - Tx, Tx - z\rangle \geq 0$ for all $x \in C$ and $z \in F(T)$.

**Lemma 2.61.** [65] *Let $T^\lambda x = Tx - \lambda\mu f(Tx)$, where $T : H \longrightarrow H$ is a nonexpansive mapping from $H$ into itself and $f$ is an $\eta$-strongly monotone and $k$-Lipschitzian mapping from $H$ into itself. If $0 \leq \lambda < 1$ and $0 < \mu < 2\eta/k^2$, then $T^\lambda$ is a contraction and satisfies*

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in H, \tag{2.9}$$

*where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$.*

**Lemma 2.62.** [48] *Let $\{a_n\}$, $\{b_n\}$ and $\delta_n$ be sequences of nonnegative real numbers satisfying the inequality,*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n, \quad n \geq 1.$$

*If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\longrightarrow\infty} a_n$ exists. If in addition, $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim_{n\longrightarrow\infty} a_n \longrightarrow 0$.*

**Lemma 2.63.** [54] *Suppose $E$ is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers $n$. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequence of $E$ such that $\limsup_{n\longrightarrow\infty} \|x_n\| \leq r$, $\limsup_{n\longrightarrow\infty} \|y_n\| \leq r$ and $\lim_{n\longrightarrow\infty} \|t_n x_n + (1 - t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0$.*

**Lemma 2.64.** [13] *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $T$ a nonexpansive mapping from $C$ into itself. If $T$ has a fixed point, then $I - T$ is demiclosed at zero, where $I$ is the identity mapping of $H$, that is, whenever $\{x_n\}$ is a sequence in $C$ weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some $y$, it follows that $(I - T)x = y$.*

**Lemma 2.65.** [11] *Let $C$ be a nonempty closed convex subset of $H$. Let $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and let $\varphi : C \longrightarrow \mathbb{R}$ be a lower semicontinuous and convex function. For $r > 0$ and $x \in H$, define a mapping $T_r : H \longrightarrow C$ as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\},$$

*for all $x \in H$. Assume that either (B1) or (B2) holds. Then, the following conclusions hold:*

1. *For each $x \in H, T_r(x) \neq \emptyset$.*
2. *$T_r$ is single-valued;*
3. *$T_r$ is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*
4. *$F(T_r) = MEP(F, \varphi)$.*
5. *$MEP(F, \varphi)$ is closed and convex.*

A Banach space $E$ is said to satisfy *Opial's condition* if for any sequence $\{x_n\}$ in $E$, $x_n \rightharpoonup x(n \to \infty)$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \forall y \in E \text{ with } x \neq y.$$

By [14, Theorem 1], it is well known that if $E$ admits a weakly sequentially continuous duality mapping, then $E$ satisfies Opial's condition, and $E$ is smooth.

We need the following lemmas for proving our main results.

**Proposition 2.66.** ([51]) *Let $E$ be a smooth banach space and let $C$ be a nonempty subset of $E$. Let $Q : E \to C$ be a retraction and let $J$ be the normalized duality mapping on $E$. Then the following are equivalent:*

*(i) $Q$ is sunny and nonexpansive;*

*(ii)* $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle, \forall x, y \in E$;

*(iii)* $\langle x - Qx, J(y - Qx)\rangle \leq 0, \forall x \in E, y \in C$.

*If* $J_q$ *is the generalized duality mapping on* $E$ *then* $\langle x - Qx, J_q(y - Qx)\rangle \leq 0, \forall x \in E, y \in C$ *is equivalent to this Proposition (see [55]).*

**Proposition 2.67.** ([12, 29, 26]) *Let* $C$ *be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space* $E$ *and let* $T$ *be a nonexpansive mapping of* $C$ *into itself with* $F(T) \neq \emptyset$. *Then the set* $F(T)$ *is a sunny nonexpansive retract of* $C$.

**Lemma 2.68.** ([3]) *Let* $C$ *be a nonempty closed convex subset of a smooth Banach space* $E$. *Let* $Q_C$ *be a sunny nonexpansive retraction from* $E$ *onto* $C$ *and let* $A$ *be an accretive operator of* $C$ *into* $E$. *Then, for all* $\lambda > 0$,

$$VI(C, A) = F(Q(I - \lambda A)),$$

*where* $VI(C, A) = \big\{x^* \in C : \langle Ax^*, J(x - x^*)\rangle \geq 0, \forall x \in C\big\}$.

**Lemma 2.69.** ([6])*Let* $C$ *be a nonempty bounded closed convex subset of a uniformly convex Banach space* $E$ *and* $T : C \to C$ *be a nonexpansive mapping. If* $\{x_n\}$ *is a sequence of* $C$ *such that* $x_n \rightharpoonup x$ *and* $x_n - Tx_n \to 0$ *then* $x$ *is a fixed point of* $T$.

**Lemma 2.70.** ([63]) *Let* $r > 0$ *and let* $E$ *be a uniformly convex Banach space. Then, there exists a continuous, strictly increasing and convex function* $g : [0, \infty) \to [0, \infty)$ *with* $g(0) = 0$ *such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

*for all* $x, y \in B_r := \{z \in E : \|z\| \leq r\}$ *and* $0 \leq \lambda \leq 1$.

**Lemma 2.71.** ([27]) *Let* $E$ *be a real smooth and uniformly convex Banach space and let* $r > 0$. *Then there exists a strictly increasing, continuous and convex function* $g : [0, 2r] \to \mathbb{R}$ *such that* $g(0) = 0$ *and*

$$g(\|x - y\|) \leq \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \forall x, y \in B_r,$$

*where* $B_r = \{z \in E : \|z\| \leq r\}$.

**Lemma 2.72.** ([63]) *Let $E$ be a real q-uniformly smooth Banach space, then there exists a constant $c_q > 0$ such that*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q, \forall x, y \in E.$$

*In particular, if $E$ is real 2-uniformly smooth Banach space, then there exists a best smooth constant $K > 0$ such that*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + 2K\|y\|^2, \forall x, y \in E.$$

**Lemma 2.73.** ([38]) *Let $E$ be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$, we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle$$

*for all $j(x + y) \in J(x + y)$ with $x \neq y$.*

**Lemma 2.74.** ([57]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $X$ and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

**Lemma 2.75.** ([64]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n,\ n \geq 0$$

*where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that*

*(1)* $\sum_{n=1}^{\infty} \alpha_n = \infty$

*(2)* $\limsup_{n\longrightarrow\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

*Then* $\lim_{n\longrightarrow\infty} a_n = 0$.

**Lemma 2.76.** ([52, 56]) *Let $C$ be a nonempty, closed and convex subset of a real $q$-uniformly smooth Banach space $E$, $L_2 : C \to E$ be a $\kappa$-Lipschitzian and $\eta$-strongly accretive operator with constants $\kappa, \eta > 0$ and let $0 < \mu < (\frac{q\eta}{c_q \kappa^q})^{\frac{1}{q-1}}$, $\tau = \mu(\eta - \frac{c_q \mu^{q-1}\kappa^q}{q})$, then for $t \in (0, \min\{1, \frac{1}{\tau}\})$, the mapping $S : C \to E$ defined by $S := (I - t\mu L_2)$ is a contraction with a constant $1 - t\tau$.*

**Lemma 2.77.** ([55]) *Let $C$ be a nonempty, closed and convex subset of a real reflexive and $q$-uniformly smooth Banach space $E$ which admits a weakly sequentially continuous generalized duality mapping $J_q$ from $E$ into $E^*$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$, $V : C \to E$ a $k$-Lipschitzian and $\eta$-strongly accretive operator with constants $k$, $\eta > 0$. Suppose $f : C \to E$ is a $L$-Lipschitzian mapping with constant $L > 0$ and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $0 < \mu < (\frac{q\eta}{c_q \kappa^q})^{\frac{1}{q-1}}$ and $0 \leq \gamma L < \tau$ where $\tau = \mu(\eta - \frac{c_q \mu^{q-1}\kappa^q}{q})$. Then $\{x_t\}$ defined by $x_t = Q_C[t\gamma f x_t + (I - t\mu V)Tx_t]$ converges strongly to some point $x^* \in F(T)$ as $t \to 0$, which is the unique solution of the variational inequality:*

$$\langle \gamma f x^* - \mu V x^*, J_q(p - x^*)\rangle \leq 0, \forall p \in F(T).$$

**Lemma 2.78.** ([55]) *Let $C$ be a closed convex subset of a smooth Banach space $E$. Let $\tilde{C}$ be a nonempty subset of $C$. Let $Q : C \to \tilde{C}$ be a retraction and let $J, J_q$ be the normalized duality mapping and generalized duality mapping on $E$, respectively. Then the following are equivalent:*

*(i) $Q$ is sunny and nonexpansive;*

*(ii) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle, \forall x, y \in E$;*

*(iii) $\langle x - Qx, J(y - Qx)\rangle \leq 0, \forall x \in C, y \in \tilde{C}$;*

*(iv) $\langle x - Qx, J_q(y - Qx)\rangle \leq 0, \forall x \in C, y \in \tilde{C}$.*

**Lemma 2.79.** ([46]) *Let $q > 1$. Then the following inequality holds:*

$$ab \leq \frac{1}{q}a^q + \frac{q-1}{q}b^{\frac{q}{q-1}}$$

*for arbitrary positive real numbers $a, b$.*

# CHAPTER III

# METHODS

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. A mapping $T : H \longrightarrow H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in H$. A mapping $f : H \longrightarrow H$ is said to be $\eta$-strongly monotone if there exists constant $\eta > 0$ such that $\langle fx - fy, x - y \rangle \geq \eta \|x - y\|^2$ for any $x, y \in H$. A mapping $f : H \longrightarrow H$ is said to be $k$-Lipschitzian if there exists a constant $k > 0$ such that $\|fx - fy\| \leq k\|x - y\|$ for any $x, y \in H$.

Let $D$ be a subset of a Hilbert space $H$. Recall that two mappings $S, T : D \longrightarrow D$ are said to satisfy condition $(A')$ which is given in [39] if there exists a nondecreasing function $f : [0, \infty) \longrightarrow [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $(1/2)(\|x - Tx\| + \|x - Sx\|) \geq f(d(x, f))$ for all $x \in D$, where $d(x, \mathcal{F}) = \inf\{\|x - x^*\| : x^* \in \mathcal{F} = F(T) \cap F(S)\}$. We modify this condition for three mappings $S, T, K : C \longrightarrow C$ as follows:

Three mappings $S, T, K : C \longrightarrow C$ where $C$ a subset of $H$, are said to satisfy condition $(A'')$ if there exists a nondecreasing function $f : [0, \infty) \longrightarrow [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $(1/3)(\|x - Tx\| + \|x - Sx\| + \|x - Kx\|) \geq f(d(x, f))$ for all $x \in C$, where $d(x, \mathcal{F}') = \inf\{\|x - x^*\| : x^* \in \mathcal{F}' = F(T) \cap F(S) \cap F(K)\}$. Note that condition $(A'')$ reduces to condition $(A')$ when $K = S$.

Let $f : H \longrightarrow H$ be a nonlinear mapping and $C$ a nonempty closed convex subset of $H$. The variational inequality problem with a mapping $f$ on $C$ ($VI(C, f)$ in short) is formulated as finding a point $u^* \in C$ such that

$$\langle f(u^*), v - u^* \rangle \geq 0, \quad \forall v \in C. \tag{2.10}$$

The variational inequalities were initially studied by Kinderlehrer and Stampacchia [41], and ever since have been widely studied. It is well known that the $VI(C,f)$ is equivalent to the fixed point equation

$$u^* = P_C(u^* - \mu f(u^*)), \tag{2.11}$$

where $P_C$ is the projection from $H$ onto $C$ and $\mu$ is an arbitrarily fixed constant. In fact, when $f$ is an $\eta$-strongly monotone and Lipschitzian mapping on $C$ and $\mu > 0$ small enough, then the mapping defined by the right hand side of (2.11) is a contraction.

For reducing the complexity of computation caused by the projection $P_C$, Yamada [65] proposed an iteration method to solve the variational inequalities $VI(C,f)$. For arbitrary $u \in H$,

$$u_{n+1} = Tu_n - \lambda_{n+1}\mu f(T(u_n)), \quad n \geq 0, \tag{2.12}$$

where $T$ is a nonexpansive mapping from $H$ into itself, $C$ is the fixed point set of $T$, $f$ is an $\eta$-strongly monotone and $k$-Lipschitzian mapping on $K$, $\{\lambda_n\}$ is a real sequence in $[0,\, 1)$, and $0 < \mu < 2\eta/k^2$. Then Yamada [65] proved that $\{u_n\}$ converges strongly to the unique solution of the $VI(C,f)$ as $\{\lambda_n\}$ satisfies the following conditions:

1. $\lim_{n\longrightarrow\infty} \lambda_n = 0$;
2. $\sum_{n=0}^{\infty} \lambda_n = \infty$;
3. $\lim_{n\longrightarrow\infty}(\lambda_n - \lambda_{n+1})/\lambda_{n+1}^2 = 0$.

Based on the idea of iterative process (2.12), recently, Wang [60] discussed the more general Mann iterative scheme as follows: Let $H$ be a Hilbert space, $T : H \longrightarrow H$ a nonexpansive mapping with $F(T) := \{x \in H, Tx = x\} \neq \emptyset$, and $f : H \longrightarrow H$ an $\eta$-strongly monotone and $k$-Lipschitzian mapping. For any

$x_0 \in H$, $\{x_n\}$ is defined by

$$x_{n+1} = a_n x_n + (1 - a_n)T^{\lambda_{n+1}} x_n, \quad \forall n \geq 0, \tag{2.13}$$

where

$$T^{\lambda} x = Tx - \lambda\mu f(Tx), \quad \forall x \in H, \tag{2.14}$$

where $\{a_n\} \subset (0,1)$ and $\{\lambda_n\} \subset [0,1)$ with some suitable conditions. Then the sequence $\{x_n\}$ is shown to converge strongly to a fixed point of $T$, and the necessary and sufficient conditions that $\{x_n\}$ converges strongly to a fixed point of $T$ are obtained.

**Theorem I** [19]. If $C$ is a compact convex subset of a Hilbert space $H, T : C \longrightarrow C$ is a Lipschitzian pseudo-contractive mapping. For $x_0 \in C$, define the sequence $\{x_n\}$ iteratively by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \quad \forall n \geq 0, \end{aligned} \tag{2.15}$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers satisfying the conditions

(i) $0 \leq \alpha_n \leq \beta_n < 1$;

(ii) $\lim_{n\longrightarrow\infty} \beta_n = 0$;

(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then the sequence $\{x_n\}$ defined by (2.15) converges strongly to a fixed point of $T$.

Let $\varphi : C \longrightarrow \mathbb{R}$ be a real-valued function and $F : C \times C \longrightarrow \mathbb{R}$ be an equilibrium bifunction, i.e., $F(u,u) = 0$ for each $u \in C$. The mixed equilibrium problem (for short, $MEP$) is to find $x^* \in C$ such that

$$MEP : F(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \forall y \in C. \tag{2.16}$$

The set of solutions for the problem $MEP$ (2.16) is denoted by $MEP(F, \varphi)$.

**Special cases.**

(1) If $\varphi \equiv 0$, then MEP (2.16) reduces to the following classical equilibrium problem (for short, $EP$):

$$\text{Finding } x^* \in C \text{ such that } F(x^*, y) \geq 0, \forall y \in C. \tag{2.17}$$

The set of solutions for the problem $EP$ (2.17) is denoted by $EP(F)$.

(2) If $\varphi \equiv 0$ and $F(x, y) = \langle Ax, y - x\rangle$ for all $x, y \in C$, where $A$ is a mapping from $C$ into $H$, then $MEP$ (2.16) reduces to the following classical variational inequality problem (for short $VIP$):

$$\text{Finding } x^* \in C \text{ such that } \langle Ax^*, y - x^*\rangle \geq 0, \forall y \in C. \tag{2.18}$$

The set of solutions for the problem $VIP$ (2.18) is denoted by $VI(C, A)$.

(3) If $F \equiv 0$, then $MEP$ (2.16) becomes the following minimize problem:

$$\text{Finding } x^* \in C \text{ such that } \varphi(y) - \varphi(x^*) \geq 0, \forall y \in C. \tag{2.19}$$

The set of solutions for the problem (2.19) is denoted by $Argmin(\varphi)$.

The problem (2.16) is very general in the sense that it includes, as special cases, fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, the equilibrium problems and others; see, e.g., [5, 7, 40] and the reference therein.

For solving the mixed equilibrium problem for an equilibrium bifunction $F : C \times C \longrightarrow \mathbb{R}$, let us assume that $F$ satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) $F$ is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;

(A3) For each $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous;

(A4) For each $x \in C$, $y \mapsto F(x, y)$ is convex;

(A5) For each $x \in C$, $y \mapsto F(x, y)$ is lower semicontinuous;

(B1) For each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subset C$ and $y_x \in C$ such that, for any $z \in C \backslash D_x$

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(B2) $C$ is a bounded set.

In this research, motivated and inspired by the above facts, we introduce a new iterative scheme for finding a common element of the set of fixed points of three nonexpansive mappings, and the set of solutions of a mixed equilibrium problem in a real Hilbert space. Strong convergence results are derived under suitable conditions in a real Hilbert space.

According to our framework throughout this research, we first preview some definitions involving a Banach space $E$ as follows. Let $U = \{x \in E : \|x\| = 1\}$.

- $E$ is said to be *uniformly convex* if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$, $\|x - y\| \geq \epsilon$ implies $\|\frac{x+y}{2}\| \leq 1 - \delta$.
  It is known that a uniformly convex Banach space is reflexive and strictly convex.

- $E$ is said to be *smooth* if $\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$ exists for all $x, y \in U$.
  It is also said to be *uniformly smooth* if the limit is attained uniformly for all $x, y \in U$. The *modulus of smoothness* of $E$ is defined by

$$\rho(\tau) = \sup\Big\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\Big\},$$

where $\rho : [0, \infty) \to [0, \infty)$ is a function.

It is known that $E$ is uniformly smooth if and only if $\lim_{\tau\to 0} \frac{\rho(\tau)}{\tau} = 0$.

- $E$ is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$ where $q$ is a fixed real number with $1 < q \leq 2$.

Let $E$ be a real Banach space and $E^*$ be the dual space of $E$ with norm $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ pairing between $E$ and $E^*$. For $q > 1$, the *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}$$

for all $x \in E$. In particular, if $q = 2$, the mapping $J_2$ is called the *normalized duality mapping* and written by $J_2 = J$ as usual. Further, we have the following properties of the generalized duality mapping $J_q$:

(i) $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \in E$ with $x \neq 0$;

(ii) $J_q(tx) = t^{q-1}J_q(x)$ for all $x \in E$ and $t \in [0, \infty)$;

(iii) $J_q(-x) = -J_q(x)$ for all $x \in E$.

Certainly, if $E$ is smooth, then $J_q$ is single-valued and can be written by $j_q$ (see also [10, 53]).

Let $C$ be a nonempty closed convex subset of a real Banach space $E$. Recall that a mapping $A : C \to C$ is said to be

(i) *Lipschitzian* with Lipschitz constant $L > 0$ if $\|Ax - Ay\| \leq L\|x - y\|, \ \forall x, y \in C$;

(ii) *nonexpansive* if $\|Ax - Ay\| \leq \|x - y\|, \ \forall x, y \in C$.

An operator $A : C \to E$ is said to be

(i) *accretive* if there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Ax - Ay, j_q(x - y)\rangle \geq 0, \ \ \forall x, y \in C;$$

(ii) *β-strongly accretive* if for any $\beta > 0$ there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Ax - Ay, j_q(x - y)\rangle \geq \beta\|x - y\|^q, \ \ \forall x, y \in C;$$

(iii) $\beta$-*inverse strongly accretive* if, for any $\beta > 0$ there exists $j_q(x-y) \in J_q(x-y)$,

$$\langle Ax - Ay, j_q(x-y)\rangle \geq \beta\|Ax - Ay\|^q, \quad \forall x, y \in C.$$

Let $D$ be a subset of $C$ and $Q : C \to D$. Then $Q$ is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$, whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A subset $D$ of $C$ is said to be a *sunny nonexpansive retract* of $C$ if there exists a sunny nonexpansive retraction $Q$ of $C$ onto $D$ (see [51, 12, 29]). A mapping $Q : C \to C$ is called a *retraction* if $Q^2 = Q$. If a mapping $Q : C \to C$ is a retraction, then $Qz = z$ for all $z$ are in the range of $Q$.

A family $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ of mappings of $C$ into itself is called a nonexpansive semigroup on $C$ if it satisfies the following conditions:

(i) $S(0)x = x$ for all $x \in C$;

(ii) $S(s + t) = S(s)S(t)$ for all $s, t \geq 0$;

(iii) $\|S(s)x - S(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;

(iv) for each $x \in C$, the mapping $S(\cdot)x$ from $[0, \infty)$ into $C$ is continuous.

Let $F(\mathcal{S})$ stands for the common fixed point set of the semigroup $\mathcal{S}$, i.e., $F(\mathcal{S}) = \{x \in C : S(s)x = x, \forall s > 0\}$. It is easy to see that $F(\mathcal{S})$ is closed and convex (see also [30, 31, 61, 15]).

In 1969, Takahashi [58] proved the first fixed point theorem for a noncommutative semigroup of nonexpansive mappings which generalizes De Marr's [44] fixed point theorem. For works related to semigroups of nonexpansive, asymptotically nonexpansive, and asymptotically nonexpansive type related to reversibility of a semigroup, we refer the reader to [18, 23, 37, 32, 33, 34, 35, 36, 59, 1, 16, 20]. In 2007, Lau et al. [35] introduced the following Mann's explicit iteration process;

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)T(\mu_n)x_n, \quad \forall n \geq 1,$$

for a semigroup $\mathcal{S} = \{T(s) : s \in S\}$ of nonexpansive mappings on a compact convex subset $C$ of a smooth and strictly convex Banach space. In 2012, Wangkeeree and Preechasilp [62] introduced the iterative scheme:

$$\begin{cases} x_1 \in C, \\ z_n = \gamma_n x_n + (1-\gamma_n)T(t_n)x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)T(t_n)z_n, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n)y_n, n \geq 0. \end{cases}$$

They proved the strong convergence theorems by using a nonexpansive semigroup in Banach spaces.

In 2006, Aoyama et al. [3] proved a weak convergence theorem in Banach spaces by using the iterative algorithm as the following

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Q_C(x_n - \lambda_n A x_n), \end{cases}$$

for all $n \geq 1$. They solved the *generalized variational inequality problem* for finding a point $x \in C$ such that

$$\langle Ax, J(y-x)\rangle \geq 0 \tag{2.20}$$

for all $y \in C$. The solution set of (2.20) is denoted by $VI(C, A)$. Variational inequality has become a rich of inspiration in pure and applied mathematics. Recently, classical variational inequality problems have been extended and generalized to study a large variety of problems arising in structural analysis, economics, optimization, operations research and engineering sciences and have witnessed an explosive growth in theoretical advances, algorithmic development, etc; see e.g. [8, 9, 21].

In 2013, Song and Ceng [55] proved a strong convergence theorem in a q-uniformly smooth Banach space as the following:

$$\begin{cases} x_1 \in C, \\ z_n = Q_C(x_n - \sigma B x_n), \\ k_n = Q_C(z_n - \lambda A z_n), \\ y_n = \beta_n k_n + (1-\alpha_n)x_n, \\ x_{n+1} = Q_C[\alpha_n \gamma f x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu V)T_n y_n], n \geq 0. \end{cases} \tag{2.21}$$

They introduced a general iterative algorithm for finding a common element of the set of common fixed points of an infinite family of nonexpansive mappings and the solution set of systems of variational inequalities.

Motivated and inspired by Wangkeeree and Preechasilp [62] and Song and Ceng [55]. In this paper, we introduce a new iterative scheme for finding common solutions of a variational inequality for an inverse-strongly accretive mapping and the solutions of a fixed point problem for a nonexpansive semigroup by using the modified Mann iterative method. We shall prove the strong convergence theorem in a $q$-uniformly smooth Banach spaces under some parameters controlling conditions. Our results extend and improve the recent results of Aoyama et al. [3], Wangkeeree and Preechasilp [62], Song and Ceng [55] and other authors.

# CHAPTER IV

# MAIN RESULTS

## Strong convergence theorem

**Theorem 3.80.** *Let $H$ be a real Hilbert space, $T, S, K : H \longrightarrow H$ a nonexpansive mapping satisfy the condition $(A'')$ with $\Omega := F(T) \cap F(S) \cap F(K) \cap MEP(F, \varphi) \neq \emptyset$. Let $f : H \longrightarrow H$ an $\eta_f$-strongly monotone and $k_f$-Lipschitzian mapping, $g : H \longrightarrow H$ an $\eta_g$-strongly monotone and $k_g$-Lipschitzian mapping, $h : H \longrightarrow H$ an $\eta_h$-strongly monotone and $k_h$-Lipschitzian mapping. For any $x_0 \in H$, $\{x_n\}$ is defined by*

$$\begin{cases} z_n = c_n x_n + (1 - c_n) K_h^{\alpha_n} T_{r_n} x_n, \\ y_n = b_n x_n + (1 - b_n) S_g^{\beta_n} z_n, \\ x_{n+1} = a_n x_n + (1 - a_n) T_f^{\lambda_{n+1}} y_n, \ \ \forall n \geq 0, \end{cases} \tag{3.22}$$

*where*

$$\begin{aligned} T_f^{\lambda_{n+1}} x &= Tx - \lambda_{n+1} \mu_f f(Tx), \quad \forall x \in H, \\ S_g^{\beta_n} x &= Sx - \beta_n \mu_g g(Sx), \quad \forall x \in H, \\ K_h^{\alpha_n} x &= Kx - \alpha_n \mu_h h(Kx), \quad \forall x \in H, \end{aligned} \tag{3.23}$$

*and $\{a_n\} \subset (0, 1)$, $\{b_n\} \subset (0, 1)$, $\{c_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset [0, 1)$, $\{\beta_n\} \subset [0, 1)$, $\{\alpha_n\} \subset [0, 1)$, $\{r_n\} \subset (0, \infty)$ satisfying the following conditions:*

*(i)* $\alpha \leq a_n \leq \beta$, $\alpha \leq b_n \leq \beta$, $\alpha \leq c_n \leq \beta$ *for some* $\alpha$, $\beta \in (0, 1)$,

*(ii)* $\sum_{n=1}^{\infty} \lambda_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ *and* $\sum_{n=1}^{\infty} \alpha_n < \infty$,

*(iii)* $0 < \mu_f < 2\eta_f / k_f^2$, $0 < \mu_g < 2\eta_g / k_g^2$ *and* $0 < \mu_h < 2\eta_h / k_h^2$,

*(iv)* $\liminf_{n \longrightarrow \infty} r_n > 0$.

*Then $\{x_n\}$ converges strongly to a point $x^* \in \Omega$.*

*Proof.* We shall show that $\{x_n\}$ is bounded.

Take $p \in$ and let $u_n = T_{r_n}x_n$. So, we have

$$\|u_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \leq \|x_n - p\|. \quad (3.24)$$

From Lemma 2.61, we have

$$\begin{aligned} \|S_g^{\beta_n}z_n - p\| &= \|S_g^{\beta_n}z_n - S_g^{\beta_n}p + S_g^{\beta_n}p - p\| \\ &\leq (1-\beta_n\tau_g)\|z_n - p\| + \beta_n\mu_g\|g(p)\|, \end{aligned} \quad (3.25)$$

$$\begin{aligned} \|K_h^{\alpha_n}u_n - p\| &= \|K_h^{\alpha_n}u_n - K_h^{\alpha_n}p + K_h^{\alpha_n}p - p\| \\ &\leq (1-\alpha_n\tau_h)\|u_n - p\| + \alpha_n\mu_h\|h(p)\| \end{aligned} \quad (3.26)$$

and

$$\begin{aligned} \|T_f^{\lambda_{n+1}}y_n - p\| &= \|T_f^{\lambda_{n+1}}y_n - T_f^{\lambda_{n+1}}p + T_f^{\lambda_{n+1}}p - p\| \\ &\leq \|T_f^{\lambda_{n+1}}y_n - T_f^{\lambda_{n+1}}p\| + \|T_f^{\lambda_{n+1}}p - p\| \\ &\leq (1-\lambda_{n+1}\tau_f)\|y_n - p\| + \lambda_{n+1}\mu_f\|f(p)\|, \end{aligned} \quad (3.27)$$

where

$$\tau_g = 1-\sqrt{1-\mu_g(2\eta_g - \mu_g k_g^2)}, \ \ \tau_f = 1-\sqrt{1-\mu_f(2\eta_f - \mu_f k_f^2)}, \ \ \tau_h = 1-\sqrt{1-\mu_h(2\eta_h - \mu_h k_h^2)}.$$

It follows that

$$\begin{aligned} \|K_h^{\alpha_n}u_n - p\|^2 &\leq (1-\alpha_n\tau_h)^2\|u_n - p\|^2 + 2(1-\alpha_n\tau_h)\alpha_n\mu_h\|h(p)\|\|u_n - p\| + \alpha_n^2\mu_h^2\|h(p)\|^2 \\ &\leq \|u_n - p\|^2 + 2\alpha_n\mu_h\|h(p)\|\|u_n - p\| + \alpha_n^2\mu_h^2\|h(p)\|^2, \end{aligned} \quad (3.28)$$

$$\begin{aligned} \|T_f^{\lambda_{n+1}}y_n - p\|^2 &\leq (1-\lambda_{n+1}\tau_f)^2\|y_n - p\|^2 + 2\lambda_{n+1}\mu_f\|f(p)\|\|y_n - p\| + \lambda_{n+1}^2\mu_f^2\|f(p)\|^2 \\ &\leq \|y_n - p\|^2 + 2\lambda_{n+1}\mu_f\|f(p)\|\|y_n - p\| + \lambda_{n+1}^2\mu_f^2\|f(p)\|^2. \end{aligned} \quad (3.29)$$

By (3.24) and (3.26), we have

$$\begin{aligned}
\|z_n - p\| &= \|c_n(x_n - p) + (1 - c_n)(K_h^{\alpha_n} u_n - p)\| \\
&\leq c_n\|x_n - p\| + (1 - c_n)(1 - \alpha_n\tau_h)\|u_n - p\| \\
&\quad + (1 - c_n)\alpha_n\mu_h\|h(p)\| \\
&\leq c_n\|x_n - p\| + (1 - c_n)(1 - \alpha_n\tau_h)\|x_n - p\| \\
&\quad + (1 - c_n)\alpha_n\mu_h\|h(p)\| \\
&\leq [c_n + (1 - c_n)(1 - \alpha_n\tau_h)]\|x_n - p\| + (1 - c_n)\alpha_n\mu_h\|h(p)\|. \quad (3.30)
\end{aligned}$$

It follows that

$$\begin{aligned}
\|z_n - p\|^2 &= \|c_n(x_n - p) + (1 - c_n)(K_h^{\alpha_n} u_n - p)\|^2 \\
&\leq c_n\|x_n - p\|^2 + (1 - c_n)\|K_h^{\alpha_n} u_n - p\|^2 \\
&\leq c_n\|x_n - p\|^2 + (1 - c_n)\Big[\|u_n - p\|^2 + 2\alpha_n\mu_h\|h(p)\|\|u_n - p\| \\
&\quad + \alpha_n^2\mu_h^2\|h(p)\|^2\Big] \\
&\leq c_n\|x_n - p\|^2 + (1 - c_n)\|u_n - p\|^2 + 2\alpha_n\mu_h\|h(p)\|\|u_n - p\| \\
&\quad + \alpha_n^2\mu_h^2\|h(p)\|^2. \quad (3.31)
\end{aligned}$$

From (3.25) and (3.30), we have

$$\begin{aligned}
\|y_n - p\| &= \|b_n(x_n - p) + (1 - b_n)(S_g^{\beta_n} z_n - p)\| \\
&\leq b_n\|x_n - p\| + (1 - b_n)(1 - \beta_n\tau_g)\|z_n - p\| + (1 - b_n)\beta_n\mu_g\|g(p)\| \\
&\leq b_n\|x_n - p\| + (1 - b_n)(1 - \beta_n\tau_g)\Big[(c_n + (1 - c_n)(1 - \alpha_n\tau_h))\|u_n - p\| \\
&\quad + (1 - c_n)\alpha_n\mu_h\|h(p)\|\Big] + (1 - b_n)\beta_n\mu_g\|g(p)\| \\
&\leq b_n\|x_n - p\| + (1 - b_n)(1 - \beta_n\tau_g)\Big[(c_n + (1 - c_n)(1 - \alpha_n\tau_h))\|x_n - p\| \\
&\quad + (1 - c_n)\alpha_n\mu_h\|h(p)\|\Big] + (1 - b_n)\beta_n\mu_g\|g(p)\| \\
&\leq [b_n + (1 - b_n)(1 - \beta_n\tau_g)c_n + (1 - b_n)(1 - \beta_n\tau_g)(1 - c_n)]\|x_n - p\| \\
&\quad + (1 - b_n)(1 - c_n)\alpha_n\mu_h\|h(p)\| + (1 - b_n)\beta_n\mu_g\|g(p)\| \\
&\leq \|x_n - p\| + \alpha_n\mu_h\|h(p)\| + \beta_n\mu_g\|g(p)\|. \quad (3.32)
\end{aligned}$$

It follows that

$$\begin{aligned}
\|y_n - p\|^2 &= \|b_n(x_n - p) + (1-b_n)(S_g^{\beta_n} z_n - p\|^2 \\
&\leq b_n\|x_n - p\|^2 + (1-b_n)\|S_g^{\beta_n} z_n - p)\|^2 \\
&\leq b_n\|x_n - p\|^2 + (1-b_n)\Big[\|z_n - p\|^2 + 2\beta_n\mu_g\|g(p)\|\|z_n - p\| \\
&\quad + \beta_n^2\mu_g^2\|g(p)\|^2\Big] \\
&\leq b_n\|x_n - p\|^2 + (1-b_n)\|z_n - p\|^2 + 2\beta_n\mu_g\|g(p)\|\|z_n - p\| \\
&\quad + \beta_n^2\mu_g^2\|g(p)\|^2. \qquad (3.33)
\end{aligned}$$

Substitute (3.30) into (3.25) to get

$$\begin{aligned}
\|S_g^{\beta_n} z_n - p\| &\leq (1-\beta_n\tau_g)\Big[[c_n + (1-c_n)(1-\alpha_n\tau_h)]\|x_n - p\| + (1-c_n)\alpha_n\mu_g\|h(p)\|\Big] \\
&\quad + \beta_n\mu_g\|g(p)\| \\
&\leq (1-\beta_n\tau_g)[c_n + (1-c_n)(1-\alpha_n\tau_h)]\|x_n - p\| \\
&\quad + (1-\beta_n\tau_g)(1-c_n)\alpha_n\mu_g\|h(p)\| + (1-\beta_n\tau_g)\beta_n\mu_g\|g(p)\| \\
&\leq (1-\beta_n\tau_g)\|x_n - p\| + \alpha_n\mu_g\|h(p)\| + \beta_n\mu_g\|g(p)\|. \qquad (3.34)
\end{aligned}$$

By (3.27), (3.30) and (3.32), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|a_n(x_n - p) + (1-a_n)(T_f^{\lambda_{n+1}} y_n - p)\| \\
&\leq a_n\|x_n - p\| + (1-a_n)\|T_f^{\lambda_{n+1}} y_n - p\| \\
&\leq a_n\|x_n - p\| + (1-a_n)(1-\lambda_{n+1}\tau_f)\|y_n - p\| + (1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \\
&\leq a_n\|x_n - p\| + (1-a_n)(1-\lambda_{n+1}\tau_f)\Big[\|x_n - p\| + \alpha_n\mu_h\|h(p)\| \\
&\quad + \beta_n\mu_g\|g(p)\|\Big] + (1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \\
&\leq a_n\|x_n - p\| + (1-a_n)(1-\lambda_{n+1}\tau_f)\|x_n - p\| \\
&\quad + (1-a_n)(1-\lambda_{n+1}\tau_f)\alpha_n\mu_h\|h(p)\| \\
&\quad + (1-a_n)(1-\lambda_{n+1}\tau_f)\beta_n\mu_g\|g(p)\| \\
&\quad + (1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \\
&\leq [a_n + (1-a_n)(1-\lambda_{n+1}\tau_f)]\|x_n - p\| + (1-a_n)\alpha_n\mu_h\|h(p)\| \\
&\quad + (1-a_n)\beta_n\mu_g\|g(p)\| + (1-a_n)\lambda_{n+1}\mu_f\|f(p) \qquad (3.35)
\end{aligned}$$

which implies that

$$\|x_{n+1} - p\| \leq \|x_n - p\| + \alpha_n \mu_h \|h(p)\| + \beta_n \mu_g \|g(p)\| + \lambda_{n+1} \mu_f \|f(p)\|. \quad (3.36)$$

From Lemma 2.62 and the conditions: $\sum_{n=1}^{\infty} \alpha_n < \infty, \sum_{n=1}^{\infty} \beta_n < \infty, \sum_{n=1}^{\infty} \lambda_n < \infty$, it follows that $\lim_{n \longrightarrow \infty} \|x_n - p\|$ exists for each $p \in \Gamma$ and $\{x_n\}$ is bounded.

Suppose that

$$\lim_{n \longrightarrow \infty} \|x_n - p\| = c \quad \text{for some} \quad c \geq 0. \quad (3.37)$$

From (3.32), we know that

$$\|y_n - p\| \leq \|x_n - p\| + \alpha_n \mu_h \|h(p)\| + \beta_n \mu_g \|g(p)\|.$$

Taking lim sup on both the sides in above inequality, we have

$$\limsup_{n \longrightarrow \infty} \|y_n - p\| \leq c. \quad (3.38)$$

Furthermore, by (3.27), we have

$$\limsup_{n \longrightarrow \infty} \|T_f^{\lambda_{n+1}} y_n - p\| \leq c. \quad (3.39)$$

Since $\lim_{n \longrightarrow \infty} \|x_{n+1} - p\| = c$, it follows that

$$\|x_{n+1} - p\| = \|a_n(x_n - p) + (1 - a_n)(T_f^{\lambda_{n+1}} y_n - p)\| \longrightarrow c$$

as $n \longrightarrow \infty$. Thus by Lemma 2.63, we have

$$\lim_{n \longrightarrow \infty} \|x_n - T_f^{\lambda_{n+1}} y_n\| = 0. \quad (3.40)$$

Next, from (3.27), we consider

$$\begin{aligned} \|x_n - p\| &\leq \|x_n - T_f^{\lambda_{n+1}} y_n\| + \|T_f^{\lambda_{n+1}} y_n - p\| \\ &\leq \|x_n - T_f^{\lambda_{n+1}} y_n\| + \|y_n - p\| + \lambda_{n+1} \mu_f \|f(p)\|, \end{aligned} \quad (3.41)$$

which implies that

$$c \leq \liminf_{n \longrightarrow \infty} \|y_n - p\| \leq \limsup_{n \longrightarrow \infty} \|y_n - p\| \leq c,$$

that is,

$$\lim_{n\longrightarrow\infty}\|y_n - p\| = \lim_{n\longrightarrow\infty}\|b_n(x_n - p) + (1 - b_n)(S_g^{\beta_n} z_n - p)\| = c. \tag{3.42}$$

From (3.34), we know that

$$\|S_g^{\beta_n} z_n - p\| \le \|x_n - p\| + \alpha_n\mu_h\|h(p)\| + \beta_n\mu_g\|g(p)\|$$

which means

$$\limsup_{n\longrightarrow\infty}\|S_g^{\beta_n} z_n - p\| \le c. \tag{3.43}$$

By Lemma 2.63, (3.42) and (3.43), we obtain

$$\lim_{n\longrightarrow\infty}\|S_g^{\beta_n} z_n - x_n\| = 0. \tag{3.44}$$

Now, by (3.30), we have

$$\begin{aligned}\|z_n - p\| &= \|c_n(x_n - p) + (1 - c_n)(K_h^{\alpha_n} u_n - p)\| \\ &\le \|x_n - p\| + (1 - c_n)\alpha_n\mu_h\|h(p)\|. \end{aligned}\tag{3.45}$$

Taking lim sup on both the sides in above inequality, we have

$$\limsup_{n\longrightarrow\infty}\|z_n - p\| \le c. \tag{3.46}$$

Next, from (3.25) and (3.44), we consider

$$\begin{aligned}\|x_n - p\| &\le \|x_n - S_g^{\beta_n} z_n\| + \|S_g^{\beta_n} z_n - p\| \\ &\le \|x_n - S_g^{\beta_n} z_n\| + \|z_n - p\| + \beta_n\mu_g\|g(p)\|, \end{aligned}\tag{3.47}$$

which implies that

$$c \le \liminf_{n\longrightarrow\infty}\|z_n - p\| \le \limsup_{n\longrightarrow\infty}\|z_n - p\| \le c,$$

i.e.

$$\lim_{n\longrightarrow\infty}\|z_n - p\| = \lim_{n\longrightarrow\infty}\|c_n(x_n - p) + (1 - c_n)(K_h^{\alpha_n} u_n - p)\| = c. \tag{3.48}$$

From (3.26), we know that

$$\|K_h^{\alpha_n} u_n - p\| \leq \|x_n - p\| + \alpha_n \mu_h \|h(p)\|$$

which means

$$\limsup_{n \longrightarrow \infty} \|K_h^{\alpha_n} u_n - p\| \leq c. \tag{3.49}$$

By Lemma 2.63, (3.48) and (3.49), we obtain

$$\lim_{n \longrightarrow \infty} \|K_h^{\alpha_n} u_n - x_n\| = 0. \tag{3.50}$$

We know that $\{x_n\}$ is bounded and $\{h(K(x_n))\}$ is bounded, thus form (3.50) it follows that

$$\begin{aligned} \|x_n - Ku_n\| &\leq \|x_n - K_h^{\alpha_n} u_n\| + \|K_h^{\alpha_n} u_n - Ku_n\| \\ &\leq \|x_n - K_h^{\alpha_n} u_n\| + \alpha_n \mu_h \|h(K(u_n))\| \longrightarrow 0. \end{aligned} \tag{3.51}$$

From (3.50), we have

$$\|z_n - x_n\| = (1 - c_n)\|K_h^{\alpha_n} u_n - x_n\| \longrightarrow 0. \tag{3.52}$$

Since $\{g(Sz_n)\}$ is bounded, by (3.44) and (3.52), it follows that

$$\begin{aligned} \|x_n - Sx_n\| &\leq \|x_n - S_g^{\beta_n} z_n\| + \|S_g^{\beta_n} z_n - Sx_n\| \\ &\leq \|x_n - S_g^{\beta_n} z_n\| + \|Sz_n - Sx_n\| + \beta_n \mu_g \|g(S(z_n))\| \\ &\leq \|x_n - S_g^{\beta_n} z_n\| + \|z_n - x_n\| + \beta_n \mu_g \|g(S(z_n))\| \longrightarrow 0. \end{aligned} \tag{3.53}$$

On the other hand, in the light of Lemma 2.65 (iii) $T_{r_n}$ is firmly nonexpansive, so we have

$$\begin{aligned} \|u_n - p\|^2 &= \|J_{r_n} x_n - J_{r_n} p\|^2 \\ &\leq \langle J_{r_n} x_n - J_{r_n} p, x_n - p \rangle = \langle u_n - p, x_n - p \rangle \\ &= \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2), \end{aligned} \tag{3.54}$$

which implies that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.55}$$

Form (3.29), (3.31), (3.33) and (3.55), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq a_n\|x_n-p\|^2+(1-a_n)\|T_f^{\lambda_{n+1}}y_n-p\|^2\\
&\leq a_n\|x_n-p\|^2+(1-a_n)\Big[\|y_n-p\|^2+2\lambda_{n+1}\mu_f\|f(p)\|\|y_n-p\|\\
&\quad+\lambda_{n+1}^2\mu_f^2\|f(p)\|^2\Big]\\
&\leq a_n\|x_n-p\|^2+(1-a_n)\|y_n-p\|^2+2\lambda_{n+1}\mu_f\|f(p)\|\|y_n-p\|\\
&\quad+\lambda_{n+1}^2\mu_f^2\|f(p)\|^2\\
&\leq a_n\|x_n-p\|^2+(1-a_n)\Big[b_n\|x_n-p\|^2+(1-b_n)\|z_n-p\|^2\\
&\quad+2\beta_n\mu_g\|g(p)\|\|z_n-p\|+\beta_n^2\mu_g^2\|g(p)\|^2\Big]\\
&\quad+2\lambda_{n+1}\mu_f\|f(p)\|\|y_n-p\|+\lambda_{n+1}^2\mu_f^2\|f(p)\|^2\\
&\leq a_n\|x_n-p\|^2+(1-a_n)b_n\|x_n-p\|^2+(1-b_n)\|z_n-p\|^2\\
&\quad+2\beta_n\mu_g\|g(p)\|\|z_n-p\|+\beta_n^2\mu_g^2\|g(p)\|^2\\
&\quad+2\lambda_{n+1}\mu_f\|f(p)\|\|y_n-p\|+\lambda_{n+1}^2\mu_f^2\|f(p)\|^2\\
&\leq a_n\|x_n-p\|^2+(1-a_n)b_n\|x_n-p\|^2\\
&\quad+(1-a_n)(1-b_n)\Big[c_n\|x_n-p\|^2+(1-c_n)\|u_n-p\|^2\\
&\quad+2\alpha_n\mu_h\|h(p)\|\|u_n-p\|+\alpha_n^2\mu_h^2\|h(p)\|^2\Big]\\
&\quad+2\lambda_{n+1}\mu_f\|f(p)\|\|y_n-p\|+\lambda_{n+1}^2\mu_f^2\|f(p)\|^2\\
&\leq a_n\|x_n-p\|^2+(1-a_n)b_n\|x_n-p\|^2+(1-a_n)(1-b_n)c_n\|x_n-p\|^2\\
&\quad+(1-a_n)(1-b_n)(1-c_n)\|u_n-p\|^2\\
&\quad+2\alpha_n\mu_h\|h(p)\|\|u_n-p\|+\alpha_n^2\mu_h^2\|h(p)\|^2\\
&\quad+2\lambda_{n+1}\mu_f\|f(p)\|\|y_n-p\|+\lambda_{n+1}^2\mu_f^2\|f(p)\|^2\\
&\leq [a_n+(1-a_n)b_n+(1-a_n)(1-b_n)c_n]\|x_n-p\|^2\\
&\quad+(1-a_n)(1-b_n)(1-c_n)\Big[\|x_n-p\|^2-\|x_n-u_n\|^2\Big]\\
&\quad+2\alpha_n\mu_h\|h(p)\|\|u_n-p\|+\alpha_n^2\mu_h^2\|h(p)\|^2\\
&\quad+2\lambda_{n+1}\mu_f\|f(p)\|\|y_n-p\|+\lambda_{n+1}^2\mu_f^2\|f(p)\|^2. \qquad (3.56)
\end{aligned}$$

It follows that

$$
\begin{aligned}
(1-a_n)(1-b_n)(1-c_n)\|x_n-u_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\quad +2\alpha_n\mu_h\|h(p)\|\|u_n-p\|+\alpha_n^2\mu_h^2\|h(p)\|^2 \\
&\quad +2\lambda_{n+1}\mu_f\|f(p)\|\|y_n-p\|+\lambda_{n+1}^2\mu_f^2\|f(p)\|^2 \\
&\leq \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|) \\
&\quad +2\alpha_n\mu_h\|h(p)\|\|u_n-p\|+\alpha_n^2\mu_h^2\|h(p)\|^2 \\
&\quad +2\lambda_{n+1}\mu_f\|f(p)\|\|y_n-p\| \\
&\quad +\lambda_{n+1}^2\mu_f^2\|f(p)\|^2. \qquad (3.57)
\end{aligned}
$$

By condition (ii), (3.37) and $\liminf_{n\longrightarrow\infty}(1-a_n)(1-b_n)(1-c_n)>0$, we obtain

$$\lim_{n\longrightarrow\infty}\|x_n-u_n\|=0 \qquad (3.58)$$

By (3.51) and (3.58), we obtain

$$
\begin{aligned}
\|x_n-Kx_n\| &\leq \|x_n-Ku_n\|+\|Ku_n-Kx_n\| \\
&\leq \|x_n-Ku_n\|+\|u_n-x_n\|\longrightarrow 0. \qquad (3.59)
\end{aligned}
$$

Moreover, from (3.40) and (3.44), it follows that

$$
\begin{aligned}
\|x_n-Tx_n\| &\leq \|Tx_n-T_f^{\lambda_{n+1}}y_n\|+\|T_f^{\lambda_{n+1}}y_n-x_n\| \\
&= \|Tx_n-[Ty_n-\lambda_{n+1}\mu_f f(T(y_n))]\|+\|T_f^{\lambda_{n+1}}y_n-x_n\| \\
&\leq \|x_n-y_n\|+\lambda_{n+1}\mu_f\|f(T(y_n))\|+\|T_f^{\lambda_{n+1}}y_n-x_n\| \\
&\leq (1-b_n)\|x_n-S_g^{\beta_n}z_n\|+\lambda_{n+1}\mu_f\|f(T(y_n))\| \\
&\quad +\|T_f^{\lambda_{n+1}}y_n-x_n\|\longrightarrow 0. \qquad (3.60)
\end{aligned}
$$

From (3.37) if $c=0$, there is nothing to prove. Suppose $c>0$. By (3.60), we know that $\lim_{n\longrightarrow\infty}\|x_n-Tx_n\|=\lim_{n\longrightarrow\infty}\|x_n-Kx_n\|=\lim_{n\longrightarrow\infty}\|x_n-Sx_n\|=0$. Since $T,S,K$ satisfy the condition $(A'')$, then $f(d(x_n,\Omega))\leq(1/3)(\|x_n-Tx_n\|+\|x_n-Sx_n\|+\|x_n-Kx_n\|)$. By (3.51), (3.53) and (3.60), we have $\lim_{n\longrightarrow\infty}f(d(x_n,\Omega))=0$. Since $f$ is a nondecreasing function and $f(0)=0$, therefore

$$\liminf_{n\longrightarrow\infty}d(x_n,\Omega)=0. \qquad (3.61)$$

For any $p \in F$, we get

$$\|f(p)\| \leq \|f(p) - f(x_n)\| + \|f(x_n)\| \leq k_f\|x_n - p\| + \|f(x_n)\|, \quad (3.62)$$

$$\|g(p)\| \leq \|g(p) - g(x_n)\| + \|g(x_n)\| \leq k_g\|x_n - p\| + \|g(x_n)\|, \quad (3.63)$$

$$\|h(p)\| \leq \|h(p) - h(x_n)\| + \|h(x_n)\| \leq k_h\|x_n - p\| + \|h(x_n)\|. \quad (3.64)$$

Note the fact that there exist two positive constants $M_1$, $M_2$, such that $\|h(x_n)\| \leq M_1$, $\|g(x_n)\| \leq M_2$ and $\|f(x_n)\| \leq M_3$. From (3.36) and the above relations, it follows that

$$\begin{aligned} \|x_{n+1} - p\| &\leq \|x_n - p\| + \alpha_n\mu_h\|h(p)\| + \beta_n\mu_g\|g(p)\| + \lambda_{n+1}\mu_f\|f(p)\| \\ &\leq (1 + \alpha_n\mu_h k_h + \beta_n\mu_g k_g + \lambda_{n+1}\mu_f k_f)\|x_n - p\| \\ &\quad + \alpha_n\mu_h\|h(p)\| + \beta_n\mu_g\|g(p)\| + \lambda_{n+1}\mu_f\|f(p)\| \\ &\leq (1 + \alpha_n\mu_h k_h + \beta_n\mu_g k_g + \lambda_{n+1}\mu_f k_f)\|x_n - p\| \\ &\quad + \alpha_n\mu_h M_1 + \beta_n\mu_g M_2 + \lambda_{n+1}\mu_f M_3. \end{aligned} \quad (3.65)$$

Thus

$$d(x_{n+1}, \Omega) \leq (1+\alpha_n\mu_h k_h+\beta_n\mu_g k_g+\lambda_{n+1}\mu_f k_f)d(x_n, F)+\alpha_n\mu_h M_1+\beta_n\mu_g M_2+\lambda_{n+1}\mu_f M_3.$$

Since $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$, by (3.61), we know that $\lim_{n\longrightarrow\infty} d(x_n, F) = 0$. We now prove that $\{x_n\}$ is a Cauchy sequence.

Taking $M = \exp(\sum_{i=0}^{\infty}(\alpha_i\mu_h k_h+\beta_i\mu_g k_g+\lambda_{i+1}\mu_f k_f))$, for any $\varepsilon > 0$, there exists positive integer $N$ such that $d(x_n, \Gamma) < \varepsilon/(2M)$ and $\sum_{i=N}^{\infty}(\alpha_i\mu_h M_1+\beta_i\mu_g M_2+\lambda_{i+1}\mu_f M_3) < \varepsilon/(2M)$ as $n \geq N$. Let $p \in \Gamma$, for any $n, m \geq N$, it follows from

(3.65) that

$$
\begin{aligned}
\|x_{n+1}-x_{m+1}\| &\leq \|x_{n+1}-p\| + \|x_{m+1}-p\| \\
&\leq (1+\alpha_n\mu_h k_h + \beta_n\mu_g k_g + \lambda_{n+1}\mu_f k_f)\|x_n - p\| + \alpha_n\mu_h M_1 + \beta_n\mu_g M_2 \\
&\quad + \lambda_{n+1}\mu_f M_3 - (1+\alpha_m\mu_h k_h + \beta_m\mu_g k_g + \lambda_{m+1}\mu_f k_f)\|x_m - p\| \\
&\quad + \alpha_m\mu_h M_1 + \beta_m\mu_g M_2 + \lambda_{m+1}\mu_f M_3 \\
&\leq \prod_{i=1}^{N}(1+\alpha_i\mu_h k_h + \beta_i\mu_g k_g + \lambda_{i+1}\mu_f k_f)\|x_N - p\| + \alpha_n\mu_h M_1 + \beta_n\mu_g M_2 \\
&\quad + \lambda_{n+1}\mu_f M_3 + \sum_{i=N}^{n-1}(\alpha_i\mu_h M_1 + \beta_i\mu_g M_2 + \lambda_{i+1}\mu_f M_3) \times \\
&\quad \prod_{j=i+1}^{n}(1+\alpha_j\mu_h k_h + \beta_j\mu_g k_g + \lambda_{j+1}\mu_f k_f) \\
&\quad + \prod_{i=N}^{m}(1+\alpha_i\mu_h k_h + \beta_i\mu_g k_g + \lambda_{i+1}\mu_f k_f)\|x_N - p\| + \alpha_m\mu_h M_1 \\
&\quad + \beta_m\mu_g M_2 + \lambda_{m+1}\mu_f M_3 + \sum_{i=N}^{m-1}(\alpha_i\mu_h M_1 + \beta_i\mu_g M_2 + \lambda_{i+1}\mu_f M_3) \times \\
&\quad \prod_{j=i+1}^{m}(1+\alpha_j\mu_h k_h + \beta_j\mu_g k_g + \lambda_{j+1}\mu_f k_f) \\
&\leq 2\exp(\sum_{i=N}^{\infty}(\alpha_i\mu_h k_h + \beta_i\mu_g k_g + \lambda_{j+1}\mu_f k_f))\|x_N - p\| \\
&\quad + 2\exp(\sum_{i=N}^{\infty}(\alpha_i\mu_h k_h + \beta_i\mu_g k_g + \lambda_{j+1}\mu_f k_f)) \\
&\quad \times \sum_{i=N}^{\infty}(\alpha_i\mu_h M_1 + \beta_i\mu_g M_2 + \lambda_{i+1}\mu_f M_3) \qquad (3.66)
\end{aligned}
$$

Thus

$$\|x_{n+1}-x_{m+1}\| \leq 2M\|x_N - p\| + 2M\sum_{i=N}^{\infty}(\alpha_i\mu_h M_1 + \beta_i\mu_g M_2 + \lambda_{i+1}\mu_f M_3),$$

which gives

$$\|x_{n+1}-x_{m+1}\| \leq 2Md(x_N, F) + 2M\sum_{i=N}^{\infty}(\alpha_i\mu_h M_1 + \beta_i\mu_g M_2 + \lambda_{i+1}\mu_f M_3) < \varepsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Therefore, there exists $x^* \in H$ such that $\{x_n\}$ converges strongly to $x^*$. It follows from $\|x_n - Tx_n\| \longrightarrow 0$ and $(I-T)$

being continuous that

$$\|(I-T)(x_n - x^*)\| \longrightarrow 0$$

as $n \longrightarrow \infty$ which implies $x^* = Tx^*$. Hence $x^* \in F(T)$. By the same reasoning, we have $x^* \in F(S)$ and $x^* \in F(K)$.

Finally, we prove that $w \in MEP(F, \varphi)$.

By $u_n = T_{r_n} x_n$, we know that

$$F(u_n, y) + \varphi(y) + \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

It follows from (A2) that

$$\varphi(y) + \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n), \quad \forall y \in C.$$

Hence,

$$\varphi(y) + \varphi(u_{n_i}) + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}), \quad \forall y \in C.$$

It follows from (A4), (A5), and the weakly lower semicontinuity of $\varphi$, $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \longrightarrow 0$ and $u_{n_i} \rightharpoonup w$ that

$$F(y, w) + \varphi(w) - \varphi(y) \leq 0, \quad \forall y \in C.$$

For $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we obtain $y_t \in C$ and hence $F(y_t, w) + \varphi(w) + \varphi(y_t) \leq 0$. So by (A4) and the convexity of $\varphi$, we have

$$\begin{aligned} 0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq tF(y_t, y) + (1-t)F(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t) \\ &\leq t[F(y_t, y) + \varphi(y) - \varphi(y_t)]. \end{aligned}$$

Dividing by $t$, we get

$$F(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0.$$

Letting $t \longrightarrow 0$, it follows from (A3) and the weakly lower semicontinuity of $\varphi$ that

$$F(w, y) + \varphi(y) - \varphi(w) \geq 0,$$

for all $y \in C$ and hence $w \in MEP(F, \varphi)$. It follows that $x^* \in \Omega$. The proof is completed □

**Corollary 3.81.** *Let $H$ be a real Hilbert space, $T, S, K : H \longrightarrow H$ a nonexpansive mapping satisfy the condition $(A'')$ with $F(T) \cap F(S) \cap F(K) \neq \emptyset$. Let $f : H \longrightarrow H$ an $\eta_f$-strongly monotone and $k_f$-Lipschitzian mapping, $g : H \longrightarrow H$ an $\eta_g$-strongly monotone and $k_g$-Lipschitzian mapping, $h : H \longrightarrow H$ an $\eta_h$-strongly monotone and $k_h$-Lipschitzian mapping. For any $x_0 \in H$, $\{x_n\}$ is defined by*

$$\begin{cases} z_n = c_n x_n + (1 - c_n) K_h^{\alpha_n} x_n, \\ y_n = b_n x_n + (1 - b_n) S_g^{\beta_n} z_n, \\ x_{n+1} = a_n x_n + (1 - a_n) T_f^{\lambda_{n+1}} y_n, \ \ \forall n \geq 0, \end{cases} \tag{3.67}$$

*where*

$$\begin{aligned} T_f^{\lambda_{n+1}} x &= Tx - \lambda_{n+1} \mu_f f(Tx), \quad \forall x \in H, \\ S_g^{\beta_n} x &= Sx - \beta_n \mu_g g(Sx), \quad \forall x \in H, \\ K_h^{\alpha_n} x &= Kx - \alpha_n \mu_h h(Kx), \quad \forall x \in H, \end{aligned} \tag{3.68}$$

*and $\{a_n\} \subset (0, 1)$, $\{b_n\} \subset (0, 1)$, $\{c_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset [0, 1)$, $\{\beta_n\} \subset [0, 1)$, $\{\alpha_n\} \subset [0, 1)$ satisfying the following conditions:*

*(i)* $\alpha \leq a_n \leq \beta$, $\alpha \leq b_n \leq \beta$, $\alpha \leq c_n \leq \beta$ *for some* $\alpha$, $\beta \in (0, 1)$,

*(ii)* $\sum_{n=1}^{\infty} \lambda_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ *and* $\sum_{n=1}^{\infty} \alpha_n < \infty$,

*(iii)* $0 < \mu_f < 2\eta_f / k_f^2$, $0 < \mu_g < 2\eta_g / k_g^2$ *and* $0 < \mu_h < 2\alpha_h / k_h^2$.

*Then $\{x_n\}$ converge strongly to a point $x^* \in F(T) \cap F(S) \cap F(K)$.*

*Proof.* Put $F(x,y) = 0$ for all $x, y \in C$, $\varphi \equiv 0$ and $r_n = 1$ in Theorem 3.80. Thus, we have $T_{r_n} x_n = x_n$. Then the sequence $\{x_n\}$ generated in Corallary 3.81 converges strongly to $x^* \in F(T) \cap F(S) \cap F(K)$. $\square$

**Theorem 3.82.** *Let $C$ be a sunny nonexpansive retract and nonempty closed convex subset of a q-uniformly smooth and uniformly convex Banach space $E$ which admits a weakly sequentially continuous generalized duality mapping $J_q : E \to E^*$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$, $A : C \to E$ be an $\beta$-inverse-strongly accretive operator, $\mathcal{S} = \{S(s) : s \geq 0\}$ be a nonexpansive semigroup from $C$ into itself, $L_1 : C \to E$ be a L-Lipschitzian mapping with constant $L \geq 0$ and $L_2 : C \to E$ be a $\kappa$-Lipschitzian and $\eta$-strongly accretive operator with constant $\kappa, \eta > 0$. Assume $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0,1)$, $\{\mu_n\} \subset (0,\infty)$ such that $\{\lambda_n\} \subset [a,b] \subset (0,1)$, $0 < \mu < (\frac{q\eta}{c_q \kappa^q})^{\frac{1}{q-1}}$ where $c_q$ is a positive real number, $0 < a \leq \lambda_n \leq b < (\frac{q\beta}{c_q})^{\frac{1}{q-1}}$, $0 \leq \gamma L < \tau$ where $\tau = \mu(\eta - \frac{c_q \mu^{q-1} \kappa^q}{q})$ and $F := F(\mathcal{S}) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and*

$$\begin{cases} z_n = Q_C(x_n - \lambda_n A x_n) \\ y_n = Q_C\big[\alpha_n \gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu L_2) S(\mu_n) z_n\big], \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S(\mu_n) y_n, \end{cases} \tag{3.69}$$

*which satisfy the following conditions:*

*(C1)* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$; *and* $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$;

*(C2)* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0, \liminf_{n\to\infty} \lambda_n > 0$;

*(C3)* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

*(C4)* $\lim_{n\to\infty} \mu_n = 0$;

*(C5)* $\lim_{n\to\infty} \sup_{x\in\tilde{C}} \|S(\mu_{n+1})x - S(\mu_n)x\| = 0$, *$\tilde{C}$ bounded subset of $C$;*

*(C6)* $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0, 0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

*Then $\{x_n\}$ converges strongly to $x^* \in F$ which also solves the following variational inequality:*

$$\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(z - x^*)\rangle \leq 0, \forall z \in F. \tag{3.70}$$

***Proof***. First of all, we prove that $\{x_n\}$ is bounded. Let $p \in F$ and $0 < a \leq \lambda_n \leq b < (\frac{q\beta}{c_q})^{\frac{1}{q-1}}$, we have

$$\begin{aligned}
\|z_n - p\|^q &= \|Q_C(x_n - \lambda_n A x_n) - Q_C(p - \lambda_n Ap)\|^q \\
&\leq \|(I - \lambda_n A)x_n - (I - \lambda_n A)p\|^q \\
&= \|(x_n - p) - \lambda_n(Ax_n - Ap)\|^q \\
&\leq \|x_n - p\|^q - q\lambda_n\langle Ax_n - Ap, j_q(x_n - p)\rangle + c_q\lambda_n^q\|Ax_n - Ap\|^q \\
&\leq \|x_n - p\|^q - q\beta\lambda_n\|Ax_n - Ap\|^2 + c_q\lambda_n^q\|Ax_n - Ap\|^q \\
&= \|x_n - p\|^q - \lambda_n(q\beta - c_q\lambda_n^{q-1})\|Ax_n - Ap\|^q \\
&\leq \|x_n - p\|^q.
\end{aligned} \tag{3.71}$$

Therefore $\|z_n - p\| \leq \|x_n - p\|$ and $I - \lambda_n A$ is a nonexpansive where $I$ is an identity mapping. By condition $(C1)$, we may assume, without loss of generality, that $\alpha_n < \min\{\alpha, \frac{\alpha}{\tau}\}$ where $0 < \alpha < \liminf_{n\to\infty}(1 - \gamma_n)$. From Lemma 2.76, we conclude that $\|(1 - \gamma_n)I - \alpha_n\mu L_2\| \leq (1 - \gamma_n) - \alpha_n\tau$. Since $0 \leq \gamma L < \tau$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n(x_n - p) + (1 - \beta_n)(S(\mu_n)y_n - p)\| \\
&\leq \beta_n\|x_n - p\| + (1 - \beta_n)\|y_n - p\| \\
&= \beta_n\|x_n - p\| + (1 - \beta_n)\big\|Q_C\big[\alpha_n\gamma L_1 x_n + \gamma_n x_n \\
&\quad +((1 - \gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n\big] - p\big\| \\
&\leq \beta_n\|x_n - p\| + (1 - \beta_n)\|[(1 - \gamma_n)I - \alpha_n\mu L_2][S(\mu_n)z_n - p] \\
&\quad +\alpha_n(\gamma L_1 x_n - \mu L_2 p) + \gamma_n(x_n - p)\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \beta_n\|x_n - p\| + (1-\beta_n)(1-\gamma_n-\alpha_n\tau)\|S(\mu_n)z_n - p\| \\
&\quad +(1-\beta_n)\alpha_n\|\gamma L_1 x_n - \mu L_2 p\| + (1-\beta_n)\gamma_n\|x_n - p\| \\
&\leq \beta_n\|x_n - p\| + (1-\beta_n)(1-\gamma_n-\alpha_n\tau)\|x_n - p\| \\
&\quad +(1-\beta_n)\alpha_n\gamma\|L_1 x_n - L_1 p\| + (1-\beta_n)\alpha_n\|\gamma L_1 p - \mu L_2 p\| \\
&\quad +(1-\beta_n)\gamma_n\|x_n - p\| \\
&\leq \beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\| - (1-\beta_n)\gamma_n\|x_n - p\| \\
&\quad -(1-\beta_n)\alpha_n\tau\|x_n - p\| + (1-\beta_n)\alpha_n\gamma L\|x_n - p\| \\
&\quad +(1-\beta_n)\alpha_n\|\gamma L_1 p - \mu L_2 p\| + (1-\beta_n)\gamma_n\|x_n - p\| \\
&= \|x_n - p\| - (1-\beta_n)\alpha_n\tau\|x_n - p\| \\
&\quad +(1-\beta_n)\alpha_n\gamma L\|x_n - p\| + (1-\beta_n)\alpha_n\|\gamma L_1 p - \mu L_2 p\| \\
&= \|x_n - p\| - (1-\beta_n)\alpha_n(\tau - \gamma L)\|x_n - p\| \\
&\quad +(1-\beta_n)\alpha_n(\tau-\gamma L)\frac{\|\gamma L_1 p - \mu L_2 p\|}{\tau - \gamma L}.
\end{aligned}
$$

By induction, we conclude that

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{\|\gamma L_1 p - \mu L_2 p\|}{\tau - \gamma L}\}, \forall n \geq 1.$$

This implies that $\{x_n\}$ is bounded, so are $\{Ax_n\}$, $\{y_n\}$, $\{S(\mu_n)y_n\}$, $\{z_n\}$ and $\{S(\mu_n)z_n\}$.

Next, we will show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and we observe that

$$
\begin{aligned}
\|z_{n+1} - z_n\| &= \|Q_C(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - Q_C(x_n - \lambda_n Ax_n)\| \\
&\leq \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_n Ax_n)\| \\
&= \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_{n+1}Ax_n) + (\lambda_n - \lambda_{n+1})Ax_n\| \\
&\leq \|(I - \lambda_{n+1}A)x_{n+1} - (I - \lambda_{n+1}A)x_n\| + |\lambda_{n+1} - \lambda_n\|Ax_n\| \\
&\leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\|,
\end{aligned}
$$

$$\begin{aligned}
\|S(\mu_{n+1})z_{n+1} - S(\mu_n)z_n\| &\leq \|S(\mu_{n+1})z_{n+1} - S(\mu_{n+1})z_n\| \\
&\quad + \|S(\mu_{n+1})z_n - S(\mu_n)z_n\| \\
&\leq \|z_{n+1} - z_n\| + \|S(\mu_{n+1})z_n - S(\mu_n)z_n\| \\
&\leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| \\
&\quad + \sup_{z \in \{z_n\}} \|S(\mu_{n+1})z - S(\mu_n)z\|,
\end{aligned}$$

and

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \big\|Q_C\big[\alpha_{n+1}\gamma L_1 x_{n+1} + \gamma_{n+1}x_{n+1} \\
&\quad + ((1-\gamma_{n+1})I - \alpha_{n+1}\mu L_2)S(\mu_{n+1})z_{n+1}\big] \\
&\quad - Q_C\big[\alpha_n\gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n\big]\big\| \\
&\leq \big\|\big[\alpha_{n+1}\gamma L_1 x_{n+1} + \gamma_{n+1}x_{n+1} \\
&\quad + ((1-\gamma_{n+1})I - \alpha_{n+1}\mu L_2)S(\mu_{n+1})z_{n+1}\big] \\
&\quad - \big[\alpha_n\gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n\big]\big\| \\
&= \big\|\big[\alpha_{n+1}\gamma L_1 x_{n+1} + \gamma_{n+1}x_{n+1} \\
&\quad + ((1-\gamma_{n+1})I - \alpha_{n+1}\mu L_2)S(\mu_{n+1})z_{n+1}\big] \\
&\quad - \big[\alpha_n\gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n\big] \\
&\quad + \alpha_{n+1}\gamma L_1 x_n - \alpha_{n+1}\gamma L_1 x_n + \gamma_{n+1}x_n - \gamma_{n+1}x_n \\
&\quad + ((1-\gamma_{n+1})I - \alpha_{n+1}\mu L_2)S(\mu_n)z_n \\
&\quad - ((1-\gamma_{n+1})I - \alpha_{n+1}\mu L_2)S(\mu_n)z_n\big\| \\
&\leq \alpha_{n+1}\gamma\|L_1 x_{n+1} - L_1 x_n\| + \gamma_{n+1}\|x_{n+1} - x_n\| \\
&\quad + \big\|\big[(1-\gamma_{n+1})I - \alpha_{n+1}\mu L_2\big]\big[S(\mu_{n+1})z_{n+1} - S(\mu_n)z_n\big]\big\| \\
&\quad + |\alpha_{n+1} - \alpha_n|\gamma\|L_1 x_n\| + |\alpha_{n+1} - \alpha_n|\mu\|L_2 S(\mu_n)z_n\| \\
&\quad + |\gamma_{n+1} - \gamma_n|\|S(\mu_n)z_n - x_n\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_{n+1}\gamma L\|x_{n+1}-x_n\| + \gamma_{n+1}\|x_{n+1}-x_n\| \\
&\quad +[(1-\gamma_{n+1})I-\alpha_{n+1}\tau]\|S(\mu_{n+1})z_{n+1}-S(\mu_n)z_n\| \\
&\quad +|\alpha_{n+1}-\alpha_n|\big[\gamma\|L_1x_n\|+\mu\|L_2S(\mu_n)z_n\|\big] \\
&\quad +|\gamma_{n+1}-\gamma_n|\|S(\mu_n)z_n-x_n\| \\
&\leq \alpha_{n+1}\gamma L\|x_{n+1}-x_n\| + \gamma_{n+1}\|x_{n+1}-x_n\| \\
&\quad +[(1-\gamma_{n+1})I-\alpha_{n+1}\tau]\big[\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\| \\
&\quad +\sup_{z\in\{z_n\}}\|S(\mu_{n+1})z-S(\mu_n)z\|\big] \\
&\quad +|\alpha_{n+1}-\alpha_n|\big[\gamma\|L_1x_n\|+\mu\|L_2S(\mu_n)z_n\|\big] \\
&\quad +|\gamma_{n+1}-\gamma_n|\|S(\mu_n)z_n-x_n\| \\
&= [1-\alpha_{n+1}(\tau-\gamma L)]\|x_{n+1}-x_n\| \\
&\quad +[(1-\gamma_{n+1})I-\alpha_{n+1}\tau]\big[|\lambda_{n+1}-\lambda_n|\|Ax_n\| \\
&\quad +\sup_{z\in\{z_n\}}\|S(\mu_{n+1})z-S(\mu_n)z\|\big] \\
&\quad +|\alpha_{n+1}-\alpha_n|\big[\gamma\|L_1x_n\|+\mu\|L_2S(\mu_n)z_n\|\big] \\
&\quad +|\gamma_{n+1}-\gamma_n|\|S(\mu_n)z_n-x_n\| \\
&\leq \|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\| \\
&\quad +\sup_{z\in\{z_n\}}\|S(\mu_{n+1})z-S(\mu_n)z\| \\
&\quad +|\alpha_{n+1}-\alpha_n|\big[\gamma\|L_1x_n\|+\mu\|L_2S(\mu_n)z_n\|\big] \\
&\quad +|\gamma_{n+1}-\gamma_n|\|S(\mu_n)z_n-x_n\| \\
&\leq \|x_{n+1}-x_n\|+\big[|\alpha_{n+1}-\alpha_n|+|\gamma_{n+1}-\gamma_n|+|\lambda_{n+1}-\lambda_n|\big]M \\
&\quad +\sup_{z\in\{z_n\}}\|S(\mu_{n+1})z-S(\mu_n)z\|,
\end{aligned}
$$

where $M=\sup_{n\geq 0}\big\{\|Ax_n\|,\gamma\|L_1x_n\|+\mu\|L_2S(\mu_n)z_n\|,\|S(\mu_n)z_n-x_n\|\big\}<\infty$. It follows that

$$\begin{aligned}
\|S(\mu_{n+1})y_{n+1} - S(\mu_n)y_n\| &\leq \|S(\mu_{n+1})y_{n+1} - S(\mu_{n+1})y_n\| + \|S(\mu_{n+1})y_n - S(\mu_n)y_n\| \\
&\leq \|y_{n+1} - y_n\| + \|S(\mu_{n+1})y_n - S(\mu_n)y_n\| \\
&\leq \|x_{n+1} - x_n\| + \big[|\alpha_{n+1} - \alpha_n| + |\gamma_{n+1} - \gamma_n| + |\lambda_{n+1} - \lambda_n|\big]M \\
&\quad + \sup_{z\in\{z_n\}} \|S(\mu_{n+1})z - S(\mu_n)z\| \\
&\quad + \sup_{y\in\{y_n\}} \|S(\mu_{n+1})y - S(\mu_n)y\|. \qquad (3.72)
\end{aligned}$$

Form the condition (C1), (C2), (C5)-(C6) and 3.72, we have

$$\limsup_{n\to\infty} \big(\|S(\mu_{n+1})y_{n+1} - S(\mu_n)y_n\| - \|x_{n+1} - x_n\|\big) \leq 0.$$

Applying Lemma 2.74, we obtain

$$\lim_{n\to\infty} \|S(\mu_n)y_n - x_n\| = 0.$$

Therefore, we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \qquad (3.73)$$

Next, we will show that $\lim_{n\to\infty} \|x_n - S(\mu_n)x_n\| = 0$, by the convexity of $\|\cdot\|^q$ for all $q > 1$, Lemma 2.72 and (3.71), we have

$$\begin{aligned}
\|y_n - p\|^q &= \|Q_C\big[\alpha_n\gamma L_1x_n + \gamma_nx_n + ((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n\big] - p\|^q \\
&\leq \|\gamma_n(x_n - p) + (1-\gamma_n)\big(S(\mu_n)z_n - p\big) + \alpha_n\big(\gamma L_1x_n - \mu L_2S(\mu_n)z_n\big)\|^q \\
&\leq \|\gamma_n(x_n - p) + (1-\gamma_n)\big(S(\mu_n)z_n - p\big)\|^q \\
&\quad + q\big\langle \alpha_n\big(\gamma L_1x_n - \mu L_2S(\mu_n)z_n\big), J_q\big(\gamma_n(x_n - p) \\
&\quad + (1-\gamma_n)\big(S(\mu_n)z_n - p\big)\big)\big\rangle \\
&\quad + c_q\|\alpha_n\big(\gamma L_1x_n - \mu L_2S(\mu_n)z_n\big)\|^q
\end{aligned}$$

$$
\begin{aligned}
&\leq \gamma_n\|x_n-p\|^q+(1-\gamma_n)\|S(\mu_n)z_n-p\|^q \\
&\quad +q\alpha_n\|\gamma L_1x_n-\mu L_2S(\mu_n)z_n\| \\
&\quad \times\|\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)\|^{q-1} \\
&\quad +c_q\alpha_n^q\|\gamma L_1x_n-\mu L_2S(\mu_n)z_n\|^q \\
&\leq \gamma_n\|x_n-p\|^q+(1-\gamma_n)\|z_n-p\|^q+\alpha_nM_1 \\
&\leq \gamma_n\|x_n-p\|^q+(1-\gamma_n)\Big[\|x_n-p\|^q-\lambda_n(q\beta-c_q\lambda_n^{q-1})\|Ax_n-Ap\|^q\Big] \\
&\quad +\alpha_nM_1 \\
&= \|x_n-p\|^q-(1-\gamma_n)\lambda_n(q\beta-c_q\lambda_n^{q-1})\|Ax_n-Ap\|^q+\alpha_nM_1,
\end{aligned}
$$

where

$$
\begin{aligned}
M_1 &= \sup_{n\geq 0}\Big\{q\|\gamma L_1x_n-\mu L_2S(\mu_n)z_n\|\|\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)\|^{q-1} \\
&\quad +c_q\alpha_n^{q-1}\|\gamma L_1x_n-\mu L_2S(\mu_n)z_n\|^q\Big\}<\infty.
\end{aligned}
$$

By the convexity of $\|\cdot\|^q$ for all $q>1$, we obtain

$$
\begin{aligned}
\|x_{n+1}-p\|^q &\leq \beta_n\|x_n-p\|^q+(1-\beta_n)\|S(\mu_n)y_n-p\|^q \\
&\leq \beta_n\|x_n-p\|^q+(1-\beta_n)\|y_n-p\|^q \\
&\leq \beta_n\|x_n-p\|^q+(1-\beta_n)\big[\|x_n-p\|^q \\
&\quad -(1-\gamma_n)\lambda_n(q\beta-c_q\lambda_n^{q-1})\|Ax-Ay\|^q+\alpha_nM_1 \\
&= \|x_n-p\|^q-(1-\beta_n)(1-\gamma_n)\lambda_n(q\beta-c_q\lambda_n^{q-1})\|Ax-Ay\|^q \\
&\quad +(1-\beta_n)\alpha_nM_1.
\end{aligned}
$$

By the fact that $a^r-b^r\leq ra^{r-1}(a-b),\forall r\geq 1$, we get
$(1-\beta_n)(1-\gamma_n)\lambda_n(q\beta-c_q\lambda_n^{q-1})\|Ax-Ay\|^q$

$$
\begin{aligned}
&\leq \|x_n-p\|^q-\|x_{n+1}-p\|^q+(1-\beta_n)\alpha_nM_1 \\
&\leq q\|x_n-p\|^{q-1}\big(\|x_n-p\|-\|x_{n+1}-p\|\big)+(1-\beta_n)\alpha_nM_1 \\
&\leq q\|x_n-p\|^{q-1}\|x_n-x_{n+1}\|+(1-\beta_n)\alpha_nM_1.
\end{aligned}
$$

From $0 < a \leq \lambda_n \leq b < (\frac{q\beta}{c_q})^{\frac{1}{q-1}}$, the conditions (C1)-(C3), (C6) and (3.73), we conclude that

$$\lim_{n\to\infty} \|Ax_n - Ap\| = 0. \tag{3.74}$$

From Proposition 2.66 (ii) and Lemma 2.71, we also have

$$\begin{aligned}
\|z_n - p\|^2 &= \|Q_C(x_n - \lambda_n Ax_n) - Q_C(p - \lambda_n Ap)\|^2 \\
&\leq \big\langle (x_n - \lambda_n Ax_n) - (p - \lambda_n Ap), J(z_n - p)\big\rangle \\
&= \big\langle (x_n - p) - \lambda_n(Ax_n - Ap), J(z_n - p)\big\rangle \\
&= \big\langle x_n - p, J(z_n - p)\big\rangle - \lambda_n\big\langle Ax_n - Ap, J(z_n - p)\big\rangle \\
&\leq \frac{1}{2}\big[\|x_n - p\|^2 + \|z_n - p\|^2 - g\|x_n - z_n\|\big] + \lambda_n\|Ax_n - Ap\|\|z_n - p\|.
\end{aligned}$$

So, we get

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - g\|x_n - z_n\| + 2\lambda_n\|Ax_n - Ap\|\|z_n - p\|.$$

By Lemma 2.73, it follows that

$$\begin{aligned}
\|y_n - p\|^2 &= \|Q_C\big[\alpha_n\gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n\big] - p\|^2 \\
&\leq \|\gamma_n(x_n - p) + (1-\gamma_n)\big(S(\mu_n)z_n - p\big) + \alpha_n\big(\gamma L_1 x_n - \mu L_2 S(\mu_n)z_n\big)\|^2 \\
&\leq \|\gamma_n(x_n - p) + (1-\gamma_n)\big(S(\mu_n)z_n - p\big)\|^2 \\
&\quad +2\alpha_n\big\langle\gamma L_1 x_n - \mu L_2 S(\mu_n)z_n, J\big(\gamma_n(x_n - p) + (1-\gamma_n)\big(S(\mu_n)z_n - p\big) \\
&\quad +\alpha_n\big(\gamma L_1 x_n - \mu L_2 S(\mu_n)z_n\big)\big)\big\rangle \\
&\leq \gamma_n\|x_n - p\|^2 + (1-\gamma_n)\|z_n - p\|^2 + \alpha_n M_2 \\
&\leq \gamma_n\|x_n - p\|^2 + (1-\gamma_n)\Big[\|x_n - p\|^2 - g\|x_n - z_n\| \\
&\quad +2\lambda_n\|Ax_n - Ap\|\|z_n - p\|\Big] + \alpha_n M_2 \\
&= \|x_n - p\|^2 - (1-\gamma_n)g\|x_n - z_n\| + 2(1-\gamma_n)\lambda_n\|Ax_n - Ap\|\|z_n - p\| \\
&\quad +\alpha_n M_2,
\end{aligned}$$

where

$$\begin{aligned}
M_2 &= \sup_{n\geq 0}\Big\{2\big\langle\gamma L_1 x_n - \mu L_2 S(\mu_n)z_n, J\big(\gamma_n(x_n - p) + (1-\gamma_n)\big(S(\mu_n)z_n - p\big) \\
&\quad +\alpha_n\big(\gamma L_1 x_n - \mu L_2 S(\mu_n)z_n\big)\big)\big\rangle\Big\} < \infty.
\end{aligned}$$

We obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\|S(\mu_n)y_n-p\|^2\\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2\\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\big[\|x_n-p\|^2-(1-\gamma_n)g\|x_n-z_n\|\\
&\quad +2(1-\gamma_n)\lambda_n\|Ax_n-Ap\|\|z_n-p\|+\alpha_nM_2\big]\\
&= \|x_n-p\|^2-(1-\beta_n)(1-\gamma_n)g\|x_n-z_n\|\\
&\quad +2(1-\beta_n)(1-\gamma_n)\lambda_n\|Ax_n-Ap\|\|z_n-p\|+(1-\beta_n)\alpha_nM_2.
\end{aligned}$$

Then we get

$$\begin{aligned}
(1-\gamma_n)g\|x_n-z_n\| &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&\quad +2(1-\beta_n)(1-\gamma_n)\lambda_n\|Ax_n-Ap\|\|z_n-p\|+(1-\beta_n)\alpha_nM_2\\
&\leq \|x_n-x_{n+1}\|\big(\|x_n-p\|+\|x_{n+1}-p\|\big)\\
&\quad +2(1-\beta_n)(1-\gamma_n)\lambda_n\|Ax_n-Ap\|\|z_n-p\|+(1-\beta_n)\alpha_nM_2.
\end{aligned}$$

By the conditions (C1)-(C3), (C6), (3.73) and (3.74), we have

$$\lim_{n\to\infty} g(\|x_n-z_n\|)=0.$$

It follows from the property of $g$ that

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.75}$$

Similar to the proof of (3.75), we start by using Lemma 2.70 and Lemma 2.73

$$
\begin{aligned}
\|y_n - p\|^2 &= \|Q_C\big[\alpha_n\gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n\big] - p\|^2 \\
&\le \|\gamma_n(x_n - p) + (1-\gamma_n)\big(S(\mu_n)z_n - p\big) + \alpha_n\big(\gamma L_1 x_n - \mu L_2 S(\mu_n)z_n\big)\|^2 \\
&\le \|\gamma_n(x_n - p) + (1-\gamma_n)\big(S(\mu_n)z_n - p\big)\|^2 \\
&\quad +2\alpha_n\big\langle \gamma L_1 x_n - \mu L_2 S(\mu_n)z_n, J\big(\gamma_n(x_n - p) + (1-\gamma_n)\big(S(\mu_n)z_n - p\big) \\
&\quad +\alpha_n\big(\gamma L_1 x_n - \mu L_2 S(\mu_n)z_n\big)\big)\big\rangle \\
&\le \gamma_n\|x_n - p\|^2 + (1-\gamma_n)\|S(\mu_n)z_n - p\|^2 \\
&\quad -\gamma_n(1-\gamma_n)g(\|x_n - S(\mu_n)z_n\|) + \alpha_n M_2 \\
&\le \gamma_n\|x_n - p\|^2 + (1-\gamma_n)\|z_n - p\|^2 \\
&\quad -\gamma_n(1-\gamma_n)g(\|x_n - S(\mu_n)z_n\|) + \alpha_n M_2 \\
&\le \gamma_n\|x_n - p\|^2 + (1-\gamma_n)\|x_n - p\|^2 \\
&\quad -\gamma_n(1-\gamma_n)g(\|x_n - S(\mu_n)z_n\|) + \alpha_n M_2 \\
&= \|x_n - p\|^2 - \gamma_n(1-\gamma_n)g(\|x_n - S(\mu_n)z_n\|) + \alpha_n M_2,
\end{aligned}
$$

where

$$
\begin{aligned}
M_2 &= \sup_{n\ge 0}\Big\{2\big\langle \gamma L_1 x_n - \mu L_2 S(\mu_n)z_n, J\big(\gamma_n(x_n - p) + (1-\gamma_n)\big(S(\mu_n)z_n - p\big) \\
&\quad +\alpha_n\big(\gamma L_1 x_n - \mu L_2 S(\mu_n)z_n\big)\big)\big\rangle\Big\} < \infty.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|S(\mu_n)y_n - p\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\big[\|x_n - p\|^2 \\
&\quad -\gamma_n(1-\gamma_n)g(\|x_n - S(\mu_n)z_n\|) + \alpha_n M_2\big] \\
&= \|x_n - p\|^2 - (1-\beta_n)\gamma_n(1-\gamma_n)g(\|x_n - S(\mu_n)z_n\|) + (1-\beta_n)\alpha_n M_2.
\end{aligned}
$$

Then we get

$$\begin{aligned}(1-\beta_n)\gamma_n(1-\gamma_n)g(\|x_n - S(\mu_n)z_n\|) &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (1-\beta_n)\alpha_n M_2 \\ &\leq \|x_n - x_{n+1}\|\big(\|x_n - p\| + \|x_{n+1} - p\|\big) \\ &\quad +(1-\beta_n)\alpha_n M_2.\end{aligned}$$

By the conditions (C1),(C3),(C6) and (3.73), we have

$$\lim_{n\to\infty} g(\|x_n - S(\mu_n)z_n\|) = 0.$$

It follows from the property of $g$ that

$$\lim_{n\to\infty} \|x_n - S(\mu_n)z_n\| = 0. \tag{3.76}$$

Since $S(\mu_n)$ is a nonexpansive and from the proof of Lemma 2.77, we get $Q_C S(\mu_n)z_n = S(\mu_n)z_n$ and observe that

$\|y_n - S(\mu_n)z_n\|$

$$\begin{aligned} &= \big\|Q_C\big[\alpha_n\gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n\big] - S(\mu_n)z_n\big\| \\ &\leq \big\|\big[\alpha_n\gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n\big] - S(\mu_n)z_n\big\| \\ &= \big\|\alpha_n\big(\gamma L_1 x_n - \mu L_2 S(\mu_n)z_n\big) + \gamma_n(x_n - S(\mu_n)z_n)\big\| \\ &\leq \alpha_n\|\gamma L_1 x_n - \mu L_2 S(\mu_n)z_n\| + \gamma_n\|x_n - S(\mu_n)z_n\|.\end{aligned}$$

It follows from the conditions (C1), (C6) and (3.76), we get

$$\lim_{n\to\infty} \|y_n - S(\mu_n)z_n\| = 0. \tag{3.77}$$

Since

$$\begin{aligned}\|x_n - S(\mu_n)x_n\| &\leq \|x_n - S(\mu_n)z_n\| + \|S(\mu_n)z_n - S(\mu_n)x_n\| \\ &\leq \|x_n - S(\mu_n)z_n\| + \|z_n - x_n\|,\end{aligned}$$

we have

$$\lim_{n\to\infty} \|x_n - S(\mu_n)x_n\| = 0.$$

Now, we show that $z \in F := F(\mathcal{S}) \cap VI(C, A)$. We can choose a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ is bounded and there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to $z$. Without loss of generality, we can assume that $x_{n_k} \rightharpoonup z$.

(I) First, we show that $z \in F(\mathcal{S})$. Let $\mu_{n_k} \geq 0$ such that $\mu_{n_k} \to 0$ and $\frac{\|S(\mu_{n_k})x_{n_k} - x_{n_k}\|}{\mu_{n_k}} \to 0, k \to \infty$. Fix $s > 0$, we can notice that

$$
\begin{aligned}
&\|x_{n_k} - S(s)z\| \\
&\leq \sum_{i=0}^{[s/\mu_{n_k}]-1} \big\|S\big((i+1)\mu_{n_k}\big)x_{n_k} - S(i\mu_{n_k})x_{n_k}\big\| \\
&\quad + \big\|S\big([s/\mu_{n_k}]\mu_k\big)x_{n_k} - S\big([s/\mu_{n_k}]\mu_{n_k}\big)z\big\| \\
&\quad + \big\|S\big([s/\mu_{n_k}]\mu_{n_k}\big)z - S(s)z\big\| \\
&\leq [s/\mu_{n_k}]\|S(\mu_{n_k})x_{n_k} - x_{n_k}\| + \|x_{n_k} - z\| + \big\|S\big(s - [s/\mu_{n_k}]\mu_{n_k}\big)z - z\big\| \\
&\leq s\frac{\|S(\mu_{n_k})x_{n_k} - x_{n_k}\|}{\mu_{n_k}} + \|x_{n_k} - z\| + \big\|S\big(s - [s/\mu_{n_k}]\mu_{n_k}\big)z - z\big\| \\
&\leq s\frac{\|S(\mu_{n_k})x_{n_k} - x_{n_k}\|}{\mu_{n_k}} + \|x_{n_k} - z\| + \max\{\|S(\mu)z - z\| : 0 \leq \mu \leq \mu_{n_k}\}.
\end{aligned}
$$

For all $k \in \mathbb{N}$, we have

$$\limsup_{k\to\infty} \|x_{n_k} - S(s)z\| \leq \limsup_{k\to\infty} \|x_{n_k} - z\|.$$

Since a Banach space $E$ with a weakly sequentially continuous duality mapping satisfies the Opial's condition, this implies $S(s)z = z$.

(II) Next, we show that $z \in VI(C, A)$. From the assumption, we see that the control sequence $\{\lambda_{n_k}\}$ is bounded. So, there exists a subsequence $\{\lambda_{n_{k_j}}\}$ converges to $\lambda_0$. We may assume, without loss of generality, that $\lambda_{n_k} \rightharpoonup \lambda_0$. Observe that

$$
\begin{aligned}
\|Q_C(x_{n_k} - \lambda_0 A x_{n_k}) - x_{n_k}\| &\leq \|Q_C(x_{n_k} - \lambda_0 A x_{n_k}) - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \\
&\leq \|(x_{n_k} - \lambda_0 A x_{n_k}) - (x_{n_k} - \lambda_{n_k} A x_{n_k})\| \\
&\quad + \|x_{n_k} - S(\mu_{n_k})z_{n_k}\| + \|S(\mu_{n_k})z_{n_k} - y_{n_k}\| \\
&\leq M\|\lambda_{n_k} - \lambda_0\| + \|x_{n_k} - S(\mu_{n_k})z_{n_k}\| \\
&\quad + \|S(\mu_{n_k})z_{n_k} - y_{n_k}\|,
\end{aligned}
$$

where $M$ is as appropriate constant such that $M \geq \sup_{n\geq 1}\{\|Ax_n\|\}$. It follows from (3.76), (3.77) and $\lambda_{n_k} \rightharpoonup \lambda_0$ that

$$\lim_{k\to\infty} \|Q_C(x_{n_k} - \lambda_0 Ax_{n_k}) - x_{n_k}\| = 0.$$

We know that $Q_C(I - \lambda_0 A)$ is nonexpansive and it follows from Lemma 2.69 that $z \in F(Q_C(I-\lambda_0 A))$. By using Lemma 2.68, we obtain that $z \in F(Q_C(I-\lambda_0 A)) = VI(C, A)$.

Therefore, from (I) and (II), we conclude that $z \in F := F(\mathcal{S}) \cap VI(C, A)$.

Next, we show that $\limsup_{n\to\infty}\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n - x^*)\rangle \leq 0$, where $x^*$ is the solution of the variational inequality (2.20). Since the Banach space $E$ has a weakly sequentially continuous generalized duality mapping $J_q : E \to E^*$ and $y_{n_k} \rightharpoonup z$, we obtain that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n - x^*)\rangle \\
&\qquad = \lim_{k\to\infty}\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_{n_k} - x^*)\rangle \\
&\qquad = \langle \gamma L_1 x^* - \mu L_2 x^*, J_q(z - x^*)\rangle \leq 0. \qquad (3.78)
\end{aligned}$$

Finally, we show that $\{x_n\}$ converges strongly to $x^*$. Setting $u_n = \alpha_n \gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu L_2)S(\mu_n)z_n, \forall n \geq 0$, it follows from Lemma 2.76, 2.78 and 2.79 that

$$\begin{aligned}
\|y_n - x^*\|^q &= \langle Q_C u_n - u_n, J_q(y_n - x^*)\rangle + \langle u_n - x^*, J_q(y_n - x^*)\rangle \\
&\leq \langle u_n - x^*, J_q(y_n - x^*)\rangle \\
&= \langle [(1-\gamma_n)I - \alpha_n \mu L_2][S(\mu_n)z_n - x^*], J_q(y_n - x^*)\rangle \\
&\quad + \alpha_n \langle \gamma L_1 x_n - \mu L_2 x^*, J_q(y_n - x^*)\rangle + \gamma_n \langle x_n - x^*, J_q(y_n - x^*)\rangle \\
&\leq (1 - \gamma_n - \alpha_n \tau)\|S(\mu_n)z_n - x^*\|\|y_n - x^*\|^{q-1} \\
&\quad + \gamma_n \|x_n - x^*\|\|y_n - x^*\|^{q-1} + \alpha_n \langle \gamma L_1 x_n - \gamma L_1 x^*, J_q(y_n - x^*)\rangle \\
&\quad + \alpha_n \langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n - x^*)\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\gamma_n-\alpha_n\tau)\|x_n-x^*\|\|y_n-x^*\|^{q-1}\\
&\quad +\gamma_n\|x_n-x^*\|\|y_n-x^*\|^{q-1}+\alpha_n\gamma L\|x_n-x^*\|\|y_n-x^*\|^{q-1}\\
&\quad +\alpha_n\langle \gamma L_1x^*-\mu L_2x^*, J_q(y_n-x^*)\rangle\\
&\leq [1-\alpha_n(\tau-\gamma L)]\|x_n-x^*\|\|y_n-x^*\|^{q-1}\\
&\quad +\alpha_n\langle \gamma L_1x^*-\mu L_2x^*, J_q(y_n-x^*)\rangle\\
&\leq [1-\alpha_n(\tau-\gamma L)]\Big[\frac{1}{q}\|x_n-x^*\|^q+\frac{q-1}{q}\|y_n-x^*\|^q\Big]\\
&\quad +\alpha_n\langle \gamma L_1x^*-\mu L_2x^*, J_q(y_n-x^*)\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|y_n-x^*\|^q &\leq \frac{1-\alpha_n(\tau-\gamma L)}{1+(q-1)\alpha_n(\tau-\gamma L)}\|x_n-x^*\|^q\\
&\quad +\frac{q\alpha_n}{1+(q-1)\alpha_n(\tau-\gamma L)}\langle \gamma L_1x^*-\mu L_2x^*, J_q(y_n-x^*)\rangle\\
&\leq [1-\alpha_n(\tau-\gamma L)]\|x_n-x^*\|^q\\
&\quad +\frac{q\alpha_n}{1+(q-1)\alpha_n(\tau-\gamma L)}\langle \gamma L_1x^*-\mu L_2x^*, J_q(y_n-x^*)\rangle.
\end{aligned}
$$

Therefore,

$\|x_{n+1}-x^*\|^q$

$$
\begin{aligned}
&\leq \beta_n\|x_n-x^*\|^q+(1-\beta_n)\|S(\mu_n)y_n-x^*\|^q\\
&\leq \beta_n\|x_n-x^*\|^q+(1-\beta_n)\|y_n-x^*\|^q\\
&\leq \beta_n\|x_n-x^*\|^q+(1-\beta_n)\Big[[1-\alpha_n(\tau-\gamma L)]\|x_n-x^*\|^q\\
&\quad +\frac{q\alpha_n}{1+(q-1)\alpha_n(\tau-\gamma L)}\langle \gamma L_1x^*-\mu L_2x^*, J_q(y_n-x^*)\rangle\Big]\\
&= [1-\alpha_n(\tau-\gamma L)(1-\beta_n)]\|x_n-p\|^q\\
&\quad +\frac{q\alpha_n(1-\beta_n)}{1+(q-1)\alpha_n(\tau-\gamma L)}\langle \gamma L_1x^*-\mu L_2x^*, J_q(y_n-x^*)\rangle. \qquad (3.79)
\end{aligned}
$$

Now, from (C1), (3.78) and applying Lemma 2.75 to (3.79), we get $\|x_n-x^*\|\to 0$ as $n\to\infty$. Therefore, the sequence $\{x_n\}$ converges strongly to $x^*\in F$. The proof is complete. □

**Corollary 3.83.** *Let $C$ be a sunny nonexpansive retract and nonempty closed convex subset of a 2-uniformly smooth and uniformly convex Banach space $E$ which admits a weakly sequentially continuous generalized duality mapping $J : E \to E^*$ with the best smooth constant $K$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$, $A : C \to E$ be an $\beta$-inverse-strongly accretive operator, $\mathcal{S} = \{S(s) : s \geq 0\}$ be a nonexpansive semigroup from $C$ into itself, $L_1 : C \to E$ be a L-Lipschitzian mapping with constant $L \geq 0$ and $L_2 : C \to E$ be a $\kappa$-Lipschitzian and $\eta$-strongly accretive operator with constant $\kappa, \eta > 0$. Assume $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0, 1)$, $\{\mu_n\} \subset (0, \infty)$ such that $\{\lambda_n\} \subset [a, b] \subset (0, 1)$, $0 < \mu < \frac{\eta}{K^2\kappa^2}$, $0 < a \leq \lambda_n \leq b < \frac{\beta}{K^2}$, $0 \leq \gamma L < \tau$ where $\tau = \mu(\eta - K^2\mu\kappa^2)$ and $F := F(\mathcal{S}) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and*

$$\begin{cases} z_n = Q_C(x_n - \lambda_n A x_n) \\ y_n = Q_C\big[\alpha_n \gamma L_1 x_n + \gamma_n x_n + ((1 - \gamma_n)I - \alpha_n \mu L_2)S(\mu_n)z_n\big], \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S(\mu_n)y_n, \end{cases}$$

*which satisfy the conditions (C1)-(C6) in Theorem 3.82. Then $\{x_n\}$ converges strongly to $x^* \in F$ which also solves the following variational inequality:*

$$\langle \gamma L_1 x^* - \mu L_2 x^*, J(z - x^*)\rangle \leq 0, \forall z \in F.$$

**Corollary 3.84.** *Let $C$ be a sunny nonexpansive retract and nonempty closed convex subset of a q-uniformly smooth and uniformly convex Banach space $E$ which admits a weakly sequentially continuous generalized duality mapping $J_q : E \to E^*$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$, $A : C \to E$ be an $\beta$-inverse-strongly accretive operator, $\mathcal{S} = \{S(s) : s \geq 0\}$ be a nonexpansive semigroup from $C$ into itself, $L_1 : C \to E$ be a L-Lipschitzian mapping with constant $L \geq 0$ and $L_2 : C \to E$ be a $\kappa$-Lipschitzian and $\eta$-strongly accretive operator with constant $\kappa, \eta > 0$. Assume $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0, 1)$, $\{\mu_n\} \subset (0, \infty)$ such that $\{\lambda_n\} \subset [a, b] \subset (0, 1)$, $0 < \mu < (\frac{q\eta}{c_q\kappa^q})^{\frac{1}{q-1}}$ where $c_q$ is a positive real number, $0 < a \leq \lambda_n \leq b < (\frac{q\beta}{c_q})^{\frac{1}{q-1}}$, $0 \leq \gamma L < \tau$ where $\tau = \mu(\eta - \frac{c_q\mu^{q-1}\kappa^q}{q})$ and*

$F := F(\mathcal{S}) \cap VI(C,A) \neq \emptyset$. *Let* $\{x_n\}$ *be the sequences defined by* $x_1 \in C$ *and*

$$\begin{cases} z_n = Q_C(x_n - \lambda_n A x_n) \\ y_n = Q_C\big[\alpha_n \gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu L_2)\frac{1}{t_n}\int_0^{t_n} S(s) z_n ds\big], \\ x_{n+1} = \beta_n x_n + (1-\beta_n)\frac{1}{t_n}\int_0^{t_n} S(s) y_n ds, \end{cases}$$

*which satisfy the conditions (C1)-(C3) and (C6) in Theorem 3.82 and assume that*

$$\lim_{n\to\infty} \sup_{x\in\tilde{C}} \left\| \frac{1}{t_{n+1}} \int_0^{t_{n+1}} S(s)x ds - \frac{1}{t_n}\int_0^{t_n} S(s)x ds \right\| = 0,$$

$\tilde{C}$ *bounded subset of* $C$, $\lim_{n\to\infty} \mu_n = \infty$ *and* $\lim_{n\to\infty} \frac{\mu_n}{\mu_{n+1}} = 1$. *Then* $\{x_n\}$ *converges strongly to* $x^* \in F$ *which also solves the following variational inequality:*

$$\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(z - x^*)\rangle \leq 0, \forall z \in F.$$

**Corollary 3.85.** *Let* $C$ *be a sunny nonexpansive retract and nonempty closed convex subset of a q-uniformly smooth and uniformly convex Banach space* $E$ *which admits a weakly sequentially continuous generalized duality mapping* $J_q : E \to E^*$. *Let* $Q_C$ *be a sunny nonexpansive retraction from* $E$ *onto* $C$, $A : C \to E$ *be an* $\beta$*-inverse-strongly accretive operator,* $L_1 : C \to E$ *be a L-Lipschitzian mapping with constant* $L \geq 0$ *and* $L_2 : C \to E$ *be a* $\kappa$*-Lipschitzian and* $\eta$*-strongly accretive operator with constant* $\kappa, \eta > 0$. *Assume* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0,1)$ *such that* $\{\lambda_n\} \subset [a,b] \subset (0,1)$, $0 < \mu < (\frac{q\eta}{c_q \kappa^q})^{\frac{1}{q-1}}$ *where* $c_q$ *is a positive real number,* $0 < a \leq \lambda_n \leq b < (\frac{q\beta}{c_q})^{\frac{1}{q-1}}$, $0 \leq \gamma L < \tau$ *where* $\tau = \mu(\eta - \frac{c_q \mu^{q-1}\kappa^q}{q})$ *and* $F := VI(C,A) \neq \emptyset$. *Let* $\{x_n\}$ *be the sequences defined by* $x_1 \in C$ *and*

$$\begin{cases} z_n = Q_C(x_n - \lambda_n A x_n) \\ y_n = Q_C\big[\alpha_n \gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu L_2) z_n\big], \\ x_{n+1} = \beta_n x_n + (1-\beta_n) y_n, \end{cases}$$

*which satisfy the conditions (C1)-(C3) and (C6) in Theorem 3.82. Then* $\{x_n\}$ *converges strongly to* $x^* \in F$ *which also solves the following variational inequality:*

$$\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(z - x^*)\rangle \leq 0, \forall z \in F.$$

*Proof.* Taking $\mu_n = 0$ in Theorem 3.82, we can conclude the desired conclusion easily. The proof is complete. □

# CHAPTER V

# CONCLUSIONS

1. Let $H$ be a real Hilbert space, $T, S, K : H \longrightarrow H$ a nonexpansive mapping satisfy the condition $(A'')$ with $\Omega := F(T) \cap F(S) \cap F(K) \cap MEP(F,\varphi) \neq \emptyset$. Let $f : H \longrightarrow H$ an $\eta_f$-strongly monotone and $k_f$-Lipschitzian mapping, $g : H \longrightarrow H$ an $\eta_g$-strongly monotone and $k_g$-Lipschitzian mapping, $h : H \longrightarrow H$ an $\eta_h$-strongly monotone and $k_h$-Lipschitzian mapping. For any $x_0 \in H$, $\{x_n\}$ is defined by

$$\begin{cases} z_n = c_n x_n + (1-c_n) K_h^{\alpha_n} T_{r_n} x_n, \\ y_n = b_n x_n + (1-b_n) S_g^{\beta_n} z_n, \\ x_{n+1} = a_n x_n + (1-a_n) T_f^{\lambda_{n+1}} y_n, \quad \forall n \geq 0, \end{cases} \tag{3.80}$$

where

$$\begin{aligned} T_f^{\lambda_{n+1}} x &= Tx - \lambda_{n+1}\mu_f f(Tx), \quad \forall x \in H, \\ S_g^{\beta_n} x &= Sx - \beta_n \mu_g g(Sx), \quad \forall x \in H, \\ K_h^{\alpha_n} x &= Kx - \alpha_n \mu_h h(Kx), \quad \forall x \in H, \end{aligned} \tag{3.81}$$

and $\{a_n\} \subset (0,1)$, $\{b_n\} \subset (0,1)$, $\{c_n\} \subset (0,1)$ and $\{\lambda_n\} \subset [0,1)$, $\{\beta_n\} \subset [0,1)$, $\{\alpha_n\} \subset [0,1)$, $\{r_n\} \subset (0,\infty)$ satisfying the following conditions:

(i) $\alpha \leq a_n \leq \beta$, $\alpha \leq b_n \leq \beta$, $\alpha \leq c_n \leq \beta$ for some $\alpha$, $\beta \in (0,1)$,

(ii) $\sum_{n=1}^{\infty} \lambda_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} \alpha_n < \infty$,

(iii) $0 < \mu_f < 2\eta_f/k_f^2$, $0 < \mu_g < 2\eta_g/k_g^2$ and $0 < \mu_h < 2\eta_h/k_h^2$,

(iv) $\liminf_{n\longrightarrow\infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to a point $x^* \in \Omega$.

2. Let $H$ be a real Hilbert space, $T, S, K : H \longrightarrow H$ a nonexpansive mapping satisfy the condition $(A'')$ with $F(T) \cap F(S) \cap F(K) \neq \emptyset$. Let $f : H \longrightarrow H$ an $\eta_f$-strongly monotone and $k_f$-Lipschitzian mapping, $g : H \longrightarrow H$ an $\eta_g$-strongly monotone and $k_g$-Lipschitzian mapping, $h : H \longrightarrow H$ an $\eta_h$-strongly monotone and $k_h$-Lipschitzian mapping. For any $x_0 \in H$, $\{x_n\}$ is defined by

$$\begin{cases} z_n = c_n x_n + (1 - c_n) K_h^{\alpha_n} x_n, \\ y_n = b_n x_n + (1 - b_n) S_g^{\beta_n} z_n, \\ x_{n+1} = a_n x_n + (1 - a_n) T_f^{\lambda_{n+1}} y_n, \ \ \forall n \geq 0, \end{cases} \tag{3.82}$$

where

$$\begin{aligned} T_f^{\lambda_{n+1}} x &= Tx - \lambda_{n+1}\mu_f f(Tx), \quad \forall x \in H, \\ S_g^{\beta_n} x &= Sx - \beta_n \mu_g g(Sx), \quad \forall x \in H, \\ K_h^{\alpha_n} x &= Kx - \alpha_n \mu_h h(Kx), \quad \forall x \in H, \end{aligned} \tag{3.83}$$

and $\{a_n\} \subset (0, 1)$, $\{b_n\} \subset (0, 1)$, $\{c_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset [0, 1)$, $\{\beta_n\} \subset [0, 1)$, $\{\alpha_n\} \subset [0, 1)$ satisfying the following conditions:

(i) $\alpha \leq a_n \leq \beta$, $\alpha \leq b_n \leq \beta$, $\alpha \leq c_n \leq \beta$ for some $\alpha, \beta \in (0, 1)$,

(ii) $\sum_{n=1}^{\infty} \lambda_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} \alpha_n < \infty$,

(iii) $0 < \mu_f < 2\eta_f / k_f^2$, $0 < \mu_g < 2\eta_g / k_g^2$ and $0 < \mu_h < 2\alpha_h / k_h^2$.

Then $\{x_n\}$ converge strongly to a point $x^* \in F(T) \cap F(S) \cap F(K)$.

3. Let $C$ be a sunny nonexpansive retract and nonempty closed convex subset of a $q$-uniformly smooth and uniformly convex Banach space $E$ which admits a weakly sequentially continuous generalized duality mapping $J_q : E \to E^*$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$, $A : C \to E$ be an $\beta$-inverse-strongly accretive operator, $\mathcal{S} = \{S(s) : s \geq 0\}$ be a nonexpansive semigroup from $C$ into itself, $L_1 : C \to E$ be a $L$-Lipschitzian mapping with constant $L \geq 0$ and

$L_2 : C \to E$ be a $\kappa$-Lipschitzian and $\eta$-strongly accretive operator with constant $\kappa, \eta > 0$. Assume $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0,1)$, $\{\mu_n\} \subset (0,\infty)$ such that $\{\lambda_n\} \subset [a,b] \subset (0,1)$, $0 < \mu < (\frac{q\eta}{c_q\kappa^q})^{\frac{1}{q-1}}$ where $c_q$ is a positive real number, $0 < a \leq \lambda_n \leq b < (\frac{q\beta}{c_q})^{\frac{1}{q-1}}$, $0 \leq \gamma L < \tau$ where $\tau = \mu(\eta - \frac{c_q\mu^{q-1}\kappa^q}{q})$ and $F := F(\mathcal{S}) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and

$$\begin{cases} z_n = Q_C(x_n - \lambda_n A x_n) \\ y_n = Q_C\big[\alpha_n\gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n\big], \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S(\mu_n)y_n, \end{cases} \tag{3.84}$$

which satisfy the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$; and $\lim_{n\to\infty}|\alpha_{n+1} - \alpha_n| = 0$;

(C2) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0, \liminf_{n\to\infty}\lambda_n > 0$;

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C4) $\lim_{n\to\infty}\mu_n = 0$;

(C5) $\lim_{n\to\infty}\sup_{x\in\tilde{C}}\|S(\mu_{n+1})x - S(\mu_n)x\| = 0$, $\tilde{C}$ bounded subset of $C$;

(C6) $\lim_{n\to\infty}|\gamma_{n+1} - \gamma_n| = 0, 0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 1$.

Then $\{x_n\}$ converges strongly to $x^* \in F$ which also solves

4. Let $C$ be a sunny nonexpansive retract and nonempty closed convex subset of a 2-uniformly smooth and uniformly convex Banach space $E$ which admits a weakly sequentially continuous generalized duality mapping $J : E \to E^*$ with the best smooth constant K. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$, $A : C \to E$ be an $\beta$-inverse-strongly accretive operator, $\mathcal{S} = \{S(s) : s \geq 0\}$ be a nonexpansive semigroup from $C$ into itself, $L_1 : C \to E$ be a $L$-Lipschitzian mapping with constant $L \geq 0$ and $L_2 : C \to E$ be a $\kappa$-Lipschitzian and $\eta$-strongly accretive operator with constant $\kappa, \eta > 0$. Assume $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0,1)$,

$\{\mu_n\} \subset (0,\infty)$ such that $\{\lambda_n\} \subset [a,b] \subset (0,1)$, $0 < \mu < \frac{\eta}{K^2\kappa^2}$, $0 < a \leq \lambda_n \leq b < \frac{\beta}{K^2}$, $0 \leq \gamma L < \tau$ where $\tau = \mu(\eta - K^2\mu\kappa^2)$ and $F := F(\mathcal{S}) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and

$$\begin{cases} z_n = Q_C(x_n - \lambda_n A x_n) \\ y_n = Q_C\big[\alpha_n \gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu L_2) S(\mu_n) z_n\big], \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S(\mu_n) y_n, \end{cases}$$

which satisfy the conditions (C1)-(C6) in Theorem 3.82. Then $\{x_n\}$ converges strongly to $x^* \in F$ which also solves the following variational inequality:

$$\langle \gamma L_1 x^* - \mu L_2 x^*, J(z - x^*)\rangle \leq 0, \forall z \in F.$$

5. Let $C$ be a sunny nonexpansive retract and nonempty closed convex subset of a $q$-uniformly smooth and uniformly convex Banach space $E$ which admits a weakly sequentially continuous generalized duality mapping $J_q : E \to E^*$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$, $A : C \to E$ be an $\beta$-inverse-strongly accretive operator, $\mathcal{S} = \{S(s) : s \geq 0\}$ be a nonexpansive semigroup from $C$ into itself, $L_1 : C \to E$ be a $L$-Lipschitzian mapping with constant $L \geq 0$ and $L_2 : C \to E$ be a $\kappa$-Lipschitzian and $\eta$-strongly accretive operator with constant $\kappa, \eta > 0$. Assume $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0,1)$, $\{\mu_n\} \subset (0,\infty)$ such that $\{\lambda_n\} \subset [a,b] \subset (0,1)$, $0 < \mu < (\frac{q\eta}{c_q\kappa^q})^{\frac{1}{q-1}}$ where $c_q$ is a positive real number, $0 < a \leq \lambda_n \leq b < (\frac{q\beta}{c_q})^{\frac{1}{q-1}}$, $0 \leq \gamma L < \tau$ where $\tau = \mu(\eta - \frac{c_q\mu^{q-1}\kappa^q}{q})$ and $F := F(\mathcal{S}) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and

$$\begin{cases} z_n = Q_C(x_n - \lambda_n A x_n) \\ y_n = Q_C\big[\alpha_n \gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu L_2) \frac{1}{t_n}\int_0^{t_n} S(s) z_n ds\big], \\ x_{n+1} = \beta_n x_n + (1-\beta_n) \frac{1}{t_n}\int_0^{t_n} S(s) y_n ds, \end{cases}$$

which satisfy the conditions (C1)-(C3) and (C6) in Theorem 3.82 and assume that

$$\lim_{n\to\infty} \sup_{x \in \tilde{C}} \left\| \frac{1}{t_{n+1}} \int_0^{t_{n+1}} S(s) x ds - \frac{1}{t_n} \int_0^{t_n} S(s) x ds \right\| = 0,$$

$\tilde{C}$ bounded subset of $C$, $\lim_{n\to\infty}\mu_n = \infty$ and $\lim_{n\to\infty}\frac{\mu_n}{\mu_{n+1}} = 1$. Then $\{x_n\}$ converges strongly to $x^* \in F$ which also solves the following variational inequality:

$$\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(z - x^*)\rangle \leq 0, \forall z \in F.$$

6. Let $C$ be a sunny nonexpansive retract and nonempty closed convex subset of a $q$-uniformly smooth and uniformly convex Banach space $E$ which admits a weakly sequentially continuous generalized duality mapping $J_q : E \to E^*$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$, $A : C \to E$ be an $\beta$-inverse-strongly accretive operator, $L_1 : C \to E$ be a $L$-Lipschitzian mapping with constant $L \geq 0$ and $L_2 : C \to E$ be a $\kappa$-Lipschitzian and $\eta$-strongly accretive operator with constant $\kappa, \eta > 0$. Assume $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0, 1)$ such that $\{\lambda_n\} \subset [a, b] \subset (0, 1)$, $0 < \mu < (\frac{q\eta}{c_q\kappa^q})^{\frac{1}{q-1}}$ where $c_q$ is a positive real number, $0 < a \leq \lambda_n \leq b < (\frac{q\beta}{c_q})^{\frac{1}{q-1}}$, $0 \leq \gamma L < \tau$ where $\tau = \mu(\eta - \frac{c_q\mu^{q-1}\kappa^q}{q})$ and $F := VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and

$$\begin{cases} z_n = Q_C(x_n - \lambda_n A x_n) \\ y_n = Q_C\big[\alpha_n\gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu L_2)z_n\big], \\ x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \end{cases}$$

which satisfy the conditions (C1)-(C3) and (C6) in Theorem 3.82. Then $\{x_n\}$ converges strongly to $x^* \in F$ which also solves the following variational inequality:

$$\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(z - x^*)\rangle \leq 0, \forall z \in F.$$

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