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สนับสนุนงบประมาณโดย

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On the study of impulsive difference equations

By

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กิตติกรรมประกาศ

รายงานวิจัยเรื่อง การศึกษาสมการผลต่างอิมพัลส์ จัดทำขึ้นเพื่อเป็นเอกสารประกอบการรายงานวิจัยสำหรับโครงการวิจัยประจำปีงบประมาณ พ.ศ.2561 มหาวิทยาลัยเทคโนโลยีราชมงคลรัตนโกสินทร์

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Abstract

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In this research project, we discuss the existence of solutions for impulsive q-difference equations with boundary conditions. Existence results are proved via fixed point theorems while the uniqueness of solutions is accomplished by means of Banach's contraction mapping principle. Examples illustrating the obtained results are also presented.

Keywords : **Existence theory, Boundary value problems, Difference equations, Impulses**

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สารบัญ

Chapter 1

Introduction

Quantum difference operators (q-difference operators) have extensive applications in diverse disciplines such as orthogonal polynomials, basic hypergeometric functions, combinatorics, the calculus of variations, mechanics and the theory of relativity. For a detailed description of such operators, we refer a text Kac and Cheung [1].

In classical quantum calculus (q-calculus), the q-derivative was first formulated by Jackson [2] in 1910 as

$$D_q x(t) = \frac{x(t) - x(qt)}{(1-q)t}, \quad 0 < q < 1, \ t \in (0, \infty). \tag{1.1}$$

The above definition does not remain valid for impulse points t_k, $k \in \mathbb{Z}$, such that $t_k \in (qt, t)$. On the other hand, this situation does not arise for impulsive equations on q-time scales as the domains consist of isolated points covering the case of consecutive points of t and qt with $t_k \notin (qt, t)$. Due to this reason, the subject of impulsive quantum difference equations on dense domains could not be studied. In [3], the authors modified the classical quantum calculus for obtaining the first and second order impulsive quantum difference equations on a dense domain $[0, T] \subset \mathbb{R}$ through the introduction of a new q-shifting operator defined by ${}_a\Phi_q(m) = qm + (1-q)a$, $m, a \in \mathbb{R}$. If $a < m$, then $a < {}_a\Phi_q(m) < m$. Let t_k, t_{k+1} be consecutive impulse points and $[t_k, t_{k+1}]$ be a dense subset of $\mathbb{R}$. For $t \in [t_k, t_{k+1}]$, we have ${}_{t_k}\Phi_q(t) \in (t_k, t_{k+1})$. The main idea was to apply quantum calculus only on a sub-interval (t_k, t_{k+1}) and then combine all intervals through impulsive conditions. In [4], the authors used the q-shifting operator to develop the new concepts of fractional quantum calculus such as the Riemann-Liouville fractional derivative and integral and their properties. They also formulated the existence and uniqueness results for some classes of first and second orders impulsive fractional q-difference equations.

Impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. Recent development in this field has been motivated by many applied problems arising in control theory, population dynamics and medicines. For some recent works on the theory of impulsive differential equations, we refer the reader to the monographs [5]-[7].

Recently, in [8], the authors applied the concepts of quantum calculus developed in [3] to study a boundary value problem of ordinary impulsive q_k-integro-difference equations of the form:

$$\begin{cases} D_{q_k}^2 x(t) = f(t, x(t), (S_{q_k}x)(t)), \quad t \in J := [0, T],\ t \neq t_k, \\ \Delta x(t_k) = I_k\left(x(t_k)\right), \quad k = 1, 2, \ldots, m, \\ D_{q_k} x(t_k^+) - D_{q_{k-1}} x(t_k) = I_k^*\left(x(t_k)\right), \quad k = 1, 2, \ldots, m, \\ x(0) + D_{q_0} x(0) = 0, \quad x(T) + D_{q_m} x(T) = 0, \end{cases} \tag{1.2}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times \mathbb{R}^{⊭} \to \mathbb{R}$, $(S_{q_k}x)(t) = \int_{t_k}^t \phi(t, s)x(s)d_{q_k}s,\ t \in (t_k, t_{k+1}],\ \ k = 0, 1, 2, \ldots, m$, $\phi : J \times J \to [0, \infty)$ is a continuous function, $I_k, I_k^* \in C(\mathbb{R}, \mathbb{R})$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$ for $k = 1, 2, \ldots, m$, $x(t_k^+) = \lim_{h \to 0} x(t_k + h)$ and $0 < q_k < 1$ for $k = 0, 1, 2, \ldots, m$. The second order q_k-difference appeared in (1.2) is defined by $D_{q_k}^2 x = D_{q_k}(D_{q_k}x)$, where the first order q_k-difference operator is

$$D_{q_k} x(t) = \frac{x(t) - x({}_{t_k}\Phi_{q_k}(t))}{(1 - q_k)(t - t_k)}, t \neq t_k, \quad D_{q_k} x(t_k) = \lim_{t \to t_k} D_{q_k} x(t). \tag{1.3}$$

Some existence and uniqueness results for problem (1.2) were proved by using a variety of fixed point theorems. More recently, the authors discussed the existence of solutions for Caputo-Hadamard type fractional impulsive hybrid systems with nonlinear fractional integral conditions in [9].

The aim of this research project is to present a new definition of Caputo type quantum difference operator and investigate the existence criteria for the solutions of an impulsive fractional q-integro-difference equation involving this operator supplemented with separated boundary conditions given by

$$\begin{cases} {}_{t_k}^{c}D_{q_k}^{\alpha_k} x(t) = f(t, x(t), {}_{t_k}I_{q_k}^{\beta_k} x(t)), \quad t \in J_k \subseteq [0, T],\ t \neq t_k, \\ \Delta x(t_k) = x(t_k^+) - x(t_k) = \varphi_k\left(x(t_k)\right), \quad k = 1, 2, \ldots, m, \\ {}_{t_k}D_{q_k} x(t_k^+) - {}_{t_{k-1}}^{c}D_{q_{k-1}}^{\gamma_{k-1}} x(t_k) = \varphi_k^*\left(x(t_k)\right), \quad k = 1, 2, \ldots, m, \\ \lambda_1 x(0) + \lambda_2\, {}_0D_{q_0} x(0) = 0, \qquad \xi_1 x(T) + \xi_2\, {}_{t_m}^{c}D_{q_m}^{\gamma_m} x(T) = 0, \end{cases} \tag{1.4}$$

where $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, ${}_{t_k}^{c}D_{q_k}^{\phi_k}$ denotes the Caputo q_k-fractional derivative of order $\phi_k \in \{\alpha_k, \gamma_k\}$ on J_k, $1 < \alpha_k \leq 2$, $0 < \gamma_k \leq 1$, $0 <$

$q_k < 1$, $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 0, 1, \dots, m$, $J = [0, T]$, $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, $\varphi_k, \varphi_k^* \in C(\mathbb{R}, \mathbb{R})$, $k = 1, 2, \dots, m$ and ${}_{t_k}I_{q_k}^{\beta_k}$ denotes the Riemann-Liouville q_k-fractional integral of order $\beta_k > 0$ on J_k, $k = 0, 1, 2, \dots, m$. The key tools to study the given problem are fixed point results due to Krasnoselskii and O'Regan which require the segregation of an operator into a sum of two operators. Some new notations of quantum constants are introduced to facilitate the process of computing.

In addition, we also study the following anti-periodic boundary value problem of impulsive fractional q-difference equation

$$\begin{cases} {}^{c}_{t_k}D_{q_k}^{\alpha_k} x(t) = f(t, x(t)), \quad t \in J_k \subseteq [0, T], \ t \neq t_k, \\ \Delta x(t_k) := x(t_k^+) - x(t_k) = \varphi_k \left({}_{t_{k-1}}I_{q_{k-1}}^{\beta_{k-1}} x(t_k) \right), \quad k = 1, 2, \dots, m, \\ {}_{t_k}D_{q_k} x(t_k^+) - {}_{t_{k-1}}D_{q_{k-1}} x(t_k) = \varphi_k^* \left({}_{t_{k-1}}I_{q_{k-1}}^{\gamma_{k-1}} x(t_k) \right), \quad k = 1, 2, \dots, m, \\ x(0) = -x(T), \qquad {}_0D_{q_0} x(0) = -{}_{t_m}D_{q_m} x(T), \end{cases} \tag{1.5}$$

where $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, ${}^{c}_{t_k}D_{q_k}^{\alpha_k}$ denotes the Caputo q_k-fractional derivative of order α_k on J_k, $1 < \alpha_k \leq 2$, $0 < q_k < 1$, $J_k = (t_k, t_{k+1}]$, $J_0 = [0, t_1]$, $k = 0, 1, \dots, m$, $J = [0, T]$, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $\varphi_k, \varphi_k^* \in C(\mathbb{R}, \mathbb{R})$, $k = 1, 2, \dots, m$, ${}_{t_k}I_{q_k}^{\beta_k}$ denotes the Riemann-Liouville q_k-fractional integral of order $\beta_k, \gamma_k > 0$ on J_k, $k = 0, 1, 2, \dots, m - 1$.

In recent years, the topic of q-calculus has attracted the attention of several researchers and a variety of new results on q-difference and fractional q-difference equations can be found in a series of books [10]-[12] and papers [13]-[31], and the references cited therein. Some applications of q-calculus have appeared in [32]-[36], but these applications do not take into account the impulsive effects. The results obtained in this research project will be useful to extend the study of these applications with impulse conditions.

Chapter 2

Basic Concepts and Preliminaries

The aim of this chapter is to give some definitions and properties of the notions of q-derivative and q-integral of the previous section on finite intervals.

2.1 Quantum Calculus on Finite Intervals

For a fixed $k \in \mathbb{N} \cup \{0\}$ let $J_k := [t_k, t_{k+1}] \subset \mathbb{R}$ be an interval and $0 < q_k < 1$ be a constant. We define q_k-derivative of a function $f : J_k \to \mathbb{R}$ at a point $t \in J_k$ as follows:

Definition 2.1.1 *Assume $f : J_k \to \mathbb{R}$ is a continuous function and let $t \in J_k$. Then the expression*

$$D_{q_k} f(t) = \frac{f(t) - f(q_k t + (1 - q_k)t_k)}{(1 - q_k)(t - t_k)}, t \neq t_k, \quad D_{q_k} f(t_k) = \lim_{t \to t_k} D_{q_k} f(t), \quad (2.1)$$

is called the q_k-derivative of function f at t.

We say that f is q_k-differentiable on J_k provided $D_{q_k} f(t)$ exists for all $t \in J_k$. Note that if $t_k = 0$ and $q_k = q$ in (2.1), then $D_{q_k} f = D_q f$, where D_q is the q-derivative of the function $f(t)$ defined in Definition **??**.

Example 2.1.2 Let $f(t) = t^2$ for $t \in [1, 4]$ and $q_k = \frac{1}{2}$. Now, we consider

$$\begin{aligned} D_{q_k} f(t) &= \frac{t^2 - (q_k t + (1 - q_k)t_k)^2}{(1 - q_k)(t - t_k)} \\ &= \frac{(1 + q_k)t^2 - 2q_k t_k t - (1 - q_k)t_k^2}{t - t_k} \\ &= \frac{3t^2 - 2t - 1}{2(t - 1)}, \ t \in (1, 4] \end{aligned}$$

and $\lim_{t\to t_k} D_{q_k} f(t) = 2$, if $t = 1$. In particular, $D_{\frac{1}{2}} f(3) = 5$ can be interpreted as a difference quotient $\dfrac{f(3) - f(2)}{3 - 2}$.

Example 2.1.3 In classical q-calculus, we have $D_q t^n = [n]_q t^{n-1}$ where $[n]_q = \dfrac{1 - q^n}{1 - q}$. However, q_k-calculus gives $D_{q_k}(t - t_k)^n = [n]_{q_k}(t - t_k)^{n-1}$. Indeed, $f(t) = (t - t_k)^n$, $t \in J_k$, then

$$\begin{aligned} D_{q_k} f(t) &= \frac{(t - t_k)^n - (q_k t + (1 - q_k)t_k - t_k)^n}{(1 - q_k)(t - t_k)} \\ &= \frac{(t - t_k)^n - q_k^n (t - t_k)^n}{(1 - q_k)(t - t_k)} \\ &= [n]_{q_k}(t - t_k)^{n-1}, \end{aligned}$$

where $[n]_{q_k} = \dfrac{1 - q_k^n}{1 - q_k}$.

Theorem 2.1.4 *[3] Assume $f, g : J_k \to \mathbb{R}$ are q_k-differentiable on J_k. Then:*

(i) The sum $f + g : J_k \to \mathbb{R}$ is q_k-differentiable on J_k with

$$D_{q_k}(f(t) + g(t)) = D_{q_k} f(t) + D_{q_k} g(t).$$

(ii) For any constant α, $\alpha f : J_k \to \mathbb{R}$ is q_k-differentiable on J_k with

$$D_{q_k}(\alpha f)(t) = \alpha D_{q_k} f(t).$$

(iii) The product $fg : J_k \to \mathbb{R}$ is q_k-differentiable on J_k with

$$\begin{aligned} D_{q_k}(fg)(t) &= f(t) D_{q_k} g(t) + g(q_k t + (1 - q_k)t_k) D_{q_k} f(t) \\ &= g(t) D_{q_k} f(t) + f(q_k t + (1 - q_k)t_k) D_{q_k} g(t). \end{aligned}$$

(iv) If $g(t)g(q_k t + (1 - q_k)t_k) \neq 0$, then $\dfrac{f}{g}$ is q_k-differentiable on J_k with

$$D_{q_k}\left(\frac{f}{g}\right)(t) = \frac{g(t) D_{q_k} f(t) - f(t) D_{q_k} g(t)}{g(t) g(q_k t + (1 - q_k)t_k)}.$$

Remark 2.1.5 In Example 2.1.3 we recall that in q-difference, if $f(t) = t^n$ then $D_q t^n = [n] t^{n-1}$. We cannot have a simple formula for q_k-difference. Using the derivative of product we have for some n :

$$\begin{aligned} D_{q_k} t &= 1, \\ D_{q_k} t^2 &= D_{q_k}(t \cdot t) = (1 + q_k)t + (1 - q_k)t_k, \\ D_{q_k} t^3 &= D_{q_k}(t^2 \cdot t) = (1 + q_k + q_k^2)t^2 + (1 + q_k - 2q_k^2)t t_k + (1 - q_k)^2 t_k^2, \\ D_{q_k} t^4 &= D_{q_k}(t^3 \cdot t) = (1 + q_k + q_k^2 + q_k^3)t^3 + (1 + q_k + q_k^2 - 3q_k^3)t_k t^2 \\ &\quad + (1 + q_k - 5q_k^2 + 3q_k^3)t_k^2 t + (1 - q_k)^3 t_k^3. \end{aligned}$$

In addition, we should define the higher q_k-derivative of functions.

Definition 2.1.6 *Let $f : J_k \to \mathbb{R}$ is a continuous function, we call the second-order q_k-derivative $D^2_{q_k} f$ provided $D_{q_k} f$ is q_k-differentiable on J_k with $D^2_{q_k} f = D_{q_k}(D_{q_k} f) : J_k \to \mathbb{R}$. Similarly, we define higher order q_k-derivative $D^n_{q_k} : J_k \to \mathbb{R}$.*

For example, if $f : J_k \to \mathbb{R}$, then we have

$$\begin{aligned} D^2_{q_k} f(t) &= D_{q_k}(D_{q_k} f(t)) \\ &= \frac{D_{q_k} f(t) - D_{q_k} f(q_k t + (1-q_k)t_k)}{(1-q_k)(t-t_k)} \\ &= \frac{\frac{f(t)-f(q_k t+(1-q_k)t_k)}{(1-q_k)(t-t_k)} - \frac{f(q_k t+(1-q_k)t_k)-f(q_k^2 t+(1-q_k^2)t_k)}{(1-q_k)(t-t_k)}}{(1-q_k)(t-t_k)} \\ &= \frac{f(t) - 2f(q_k t + (1-q_k)t_k) + f(q_k^2 t + (1-q_k^2)t_k)}{(1-q_k)^2(t-t_k)^2}, \quad t \neq t_k, \end{aligned}$$

and $D^2_{q_k} f(t_k) = \lim_{t \to t_k} D^2_{q_k} f(t)$.

To construct the q_k-antiderivative $F(t)$, we define a shifting operator by

$$E_{q_k} F(t) = F(q_k t + (1-q_k)t_k).$$

It is easy to prove by using mathematical induction that

$$E^n_{q_k} F(t) = E_{q_k}(E^{n-1}_{q_k} F)(t) = F(q_k^n t + (1-q_k^n)t_k),$$

where $n \in \mathbb{N}$ and $E^0_{q_k} F(t) = F(t)$.
Then we have by Definition 2.1.1 that

$$\frac{F(t) - F(q_k t + (1-q_k)t_k)}{(1-q_k)(t-t_k)} = \frac{1 - E_{q_k}}{(1-q_k)(t-t_k)} F(t) = f(t).$$

Therefore, the q_k-antiderivative can be expressed as

$$F(t) = \frac{1}{1 - E_{q_k}} \left((1-q_k)(t-t_k)f(t)\right).$$

Using the geometric series expansion, we obtain

$$\begin{aligned} F(t) &= (1-q_k) \sum_{n=0}^{\infty} E^n_{q_k} (t-t_k) f(t) \\ &= (1-q_k) \sum_{n=0}^{\infty} \left(q_k^n t + (1-q_k^n)t_k - t_k\right) f(q_k^n t + (1-q_k^n)t_k) \\ &= (1-q_k)(t-t_k) \sum_{n=0}^{\infty} q_k^n f(q_k^n t + (1-q_k^n)t_k). \end{aligned} \tag{2.2}$$

It is clear that the above calculus is valid only if the series in the right-hand side of (2.2) is convergent.

Definition 2.1.7 *Assume $f : J_k \to \mathbb{R}$ is a continuous function. Then the q_k-integral is defined by*

$$\int_{t_k}^{t} f(s)d_{q_k}s = (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f(q_k^n t + (1-q_k^n)t_k) \tag{2.3}$$

for $t \in J_k$. Moreover, if $a \in (t_k, t)$ then the definite q_k-integral is defined by

$$\begin{aligned}\int_{a}^{t} f(s)d_{q_k}s &= \int_{t_k}^{t} f(s)d_{q_k}s - \int_{t_k}^{a} f(s)d_{q_k}s \\ &= (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f(q_k^n t + (1-q_k^n)t_k) \\ &\quad -(1-q_k)(a-t_k)\sum_{n=0}^{\infty} q_k^n f(q_k^n a + (1-q_k^n)t_k).\end{aligned}$$

Note that if $t_k = 0$ and $q_k = q$, then (2.3) reduces to q-integral of a function $f(t)$, defined by $\int_{0}^{t} f(s)d_q s = (1-q)t\sum_{n=0}^{\infty} q^n f(q^n t)$ for $t \in [0, \infty)$.

Example 2.1.8 Let $f(t) = t$ for $t \in J_k$, then we have

$$\begin{aligned}\int_{t_k}^{t} f(s)d_{q_k}s &= \int_{t_k}^{t} s d_{q_k}s \\ &= (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n (q_k^n t + (1-q_k^n)t_k) \\ &= \frac{(t-t_k)(t+q_k t_k)}{1+q_k}.\end{aligned}$$

Theorem 2.1.9 *[3] For $t \in J_k$, the following formulas hold:*

(i) $D_{q_k}\int_{t_k}^{t} f(s)d_{q_k}s = f(t)$;

(ii) $\int_{t_k}^{t} D_{q_k}f(s)d_{q_k}s = f(t)$;

(iii) $\int_{a}^{t} D_{q_k}f(s)d_{q_k}s = f(t) - f(a)$ *for $a \in (t_k, t)$.*

Theorem 2.1.10 *[3] Assume $f, g : J_k \to \mathbb{R}$ are continuous functions, $\alpha \in \mathbb{R}$. Then, for $t \in J_k$,*

(i) $\displaystyle\int_{t_k}^{t} [f(s) + g(s)] d_{q_k} s = \int_{t_k}^{t} f(s) d_{q_k} s + \int_{t_k}^{t} g(s) d_{q_k} s;$

(ii) $\displaystyle\int_{t_k}^{t} (\alpha f)(s) d_{q_k} s = \alpha \int_{t_k}^{t} f(s) d_{q_k} s;$

(iii) $\displaystyle\int_{t_k}^{t} f(s) D_{q_k} g(s) d_{q_k} s = (fg)(t) - \int_{t_k}^{t} g(q_k s + (1 - q_k) t_k) D q_k f(s) d_{q_k} s.$

Theorem 2.1.11 *(Reversing the order of q_k-integration)[3]. Let $f \in C(J_k, \mathbb{R})$, then the following formula holds*

$$\int_{t_k}^{t} \int_{t_k}^{s} f(r) d_{q_k} r d_{q_k} s = \int_{t_k}^{t} \int_{q_k r + (1 - q_k) t_k}^{t} f(r) d_{q_k} s d_{q_k} r.$$

2.2 Fractional Quantum Calculus on Finite Intervals

This section is devoted to some basic concepts such as q-shifting operator, Riemann-Liouville fractional q-integral and q-difference on a given interval [4].

We define a q-shifting operator as

$$_a\Phi_q(m) = qm + (1 - q)a. \tag{2.4}$$

For any positive integer k, we have

$$_a\Phi_q^k(m) = {_a\Phi_q^{k-1}} ({_a\Phi_q}(m)) \quad \text{and} \quad {_a\Phi_q^0}(m) = m.$$

The power of q-shifting operator is defined as

$$_a(n - m)_q^{(0)} = 1, \quad {_a(n - m)_q^{(k)}} = \prod_{i=0}^{k-1} \left(n - {_a\Phi_q^i}(m)\right), \quad k \in \mathbb{N} \cup \{\infty\}.$$

More generally, if $\gamma \in \mathbb{R}$, then

$$_a(n - m)_q^{(\gamma)} = n^{(\gamma)} \prod_{i=0}^{\infty} \frac{1 - {_{\frac{a}{n}}\Phi_q^i}(m/n)}{1 - {_{\frac{a}{n}}\Phi_q^{\gamma+i}}(m/n)}.$$

The q-derivative of a function f on interval $[a, b]$ is defined by

$$({_aD_q} f)(t) = \frac{f(t) - f({_a\Phi_q}(t))}{(1 - q)(t - a)}, \quad t \neq a, \quad \text{and} \quad ({_aD_q} f)(a) = \lim_{t \to a} ({_aD_q} f)(t),$$

and q-derivative of higher order are given by

$$({}_aD_q^0 f)(t) = f(t) \text{ and } ({}_aD_q^k f)(t) = {}_aD_q^{k-1}({}_aD_q f)(t), \quad k \in \mathbb{N}.$$

The q-derivative of a product and ratio of functions f and g on $[a,b]$ are

$$\begin{aligned} {}_aD_q(fg)(t) &= f(t)_aD_q g(t) + g({}_a\Phi_q(t))_aD_q f(t) \\ &= g(t)_aD_q f(t) + f({}_a\Phi_q(t))_aD_q g(t), \end{aligned}$$

and

$${}_aD_q\left(\frac{f}{g}\right)(t) = \frac{g(t)_aD_q f(t) - f(t)_aD_q g(t)}{g(t)g({}_a\Phi_q(t))},$$

where $g(t)g({}_a\Phi_q(t)) \neq 0$.

The q-integral of a function f defined on the interval $[a,b]$ is given by

$$({}_aI_q f)(t) = \int_a^t f(s)_a ds = (1-q)(t-a)\sum_{i=0}^{\infty} q^i f({}_a\Phi_{q^i}(t)), \quad t \in [a,b], \tag{2.5}$$

with

$$({}_aI_q^0 f)(t) = f(t) \text{ and } ({}_aI_q^k f)(t) = {}_aI_q^{k-1}({}_aI_q f)(t), \quad k \in \mathbb{N}.$$

The fundamental theorem of calculus applies to the operator ${}_aD_q$ and ${}_aI_q$, that is,

$$({}_aD_{qa}I_q f)(t) = f(t),$$

and if f is continuous at $t=a$, then

$$({}_aI_{qa}D_q f)(t) = f(t) - f(a).$$

The formula for q-integration by parts on interval $[a,b]$ is

$$\int_a^b f(s)_aD_q g(s)_a d_q s = (fg)(t)\Big|_a^b - \int_a^b g({}_a\Phi_q(s))_aD_q f(s)_a d_q s.$$

Let us now define Riemann-Liouville fractional q-derivative and q-integral on interval $[a,b]$ and outline some of their properties [4].

Definition 2.2.1 *The fractional q-derivative of Riemann-Liouville type of order $\nu \geq 0$ on interval $[a,b]$ is defined by $({}_aD_q^0 f)(t) = f(t)$ and*

$$({}_aD_q^\nu f)(t) = ({}_aD_{qa}^l I_q^{l-\nu} f)(t), \quad \nu > 0,$$

where l is the smallest integer greater than or equal to ν.

Definition 2.2.2 *Let* $\alpha \geq 0$ *and* f *be a function defined on* $[a,b]$*. The fractional* q*-integral of Riemann-Liouville type is given by* $({}_aI_q^0 f)(t) = h(t)$ *and*

$$({}_aI_q^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t {}_a(t - {}_a\Phi_q(s))_q^{(\alpha-1)} f(s)\, {}_ad_qs, \quad \alpha > 0, \quad t \in [a,b].$$

From [4], we have the following formulas

$${}_aD_q^\alpha (t-a)^\beta = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, \tag{2.6}$$

$${}_aI_q^\alpha (t-a)^\beta = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}(t-a)^{\beta+\alpha}. \tag{2.7}$$

Lemma 2.2.3 *Let* $\alpha,\ \beta \in \mathbb{R}^+$ *and* f *be a continuous function on* $[a,b]$*,* $a \geq 0$*. The Riemann-Liouville fractional* q*-integral has the following semi-group property*

$${}_aI_q^\beta\, {}_aI_q^\alpha f(t) = {}_aI_q^\alpha\, {}_aI_q^\beta f(t) = {}_aI_q^{\alpha+\beta} f(t).$$

Lemma 2.2.4 *Let* f *be a* q*-integrable function on* $[a,b]$*. Then the following equality holds*

$${}_aD_q^\alpha\, {}_aI_q^\alpha f(t) = f(t), \qquad \text{for} \;\; \alpha > 0,\ t \in [a,b].$$

Lemma 2.2.5 *Let* $\alpha > 0$ *and* p *be a positive integer. Then for* $t \in [a,b]$ *the following equality holds*

$${}_aI_q^\alpha\, {}_aD_q^p f(t) = {}_aD_q^p\, {}_aI_q^\alpha f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)}\, {}_aD_q^k f(a).$$

We define Caputo fractional q-derivative as follows.

Definition 2.2.6 *The fractional* q*-derivative of Caputo type of order* $\alpha \geq 0$ *on interval* $[a,b]$ *is defined by* $({}_a^cD_q^0 f)(t) = f(t)$ *and*

$$({}_a^cD_q^\alpha f)(t) = ({}_aI_q^{n-\alpha}\, {}_aD_q^n f)(t), \quad \alpha > 0,$$

where n *is the smallest integer greater than or equal to* α*.*

Lemma 2.2.7 *Let* $\alpha > 0$ *and* n *be the smallest integer greater than or equal to* α*. Then for* $t \in [a,b]$ *the following equality holds*

$${}_aI_q^\alpha\, {}_a^cD_q^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{\Gamma_q(k+1)}\, {}_aD_q^k f(a).$$

Proof. From Lemma 2.2.5, for $\alpha = p = m$, where m is a positive integer, we have

$$\begin{aligned} {}_aI_q^m {}_aD_q^m f(t) &= {}_aD_q^m {}_aI_q^m f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^k}{\Gamma_q(k+1)} {}_aD_q^k f(a) \\ &= f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^k}{\Gamma_q(k+1)} {}_aD_q^k f(a). \end{aligned}$$

Then, by Definition 2.2.6, we have

$$\begin{aligned} {}_aI_q^{\alpha c} {}_aD_q^{\alpha} f(t) &= {}_aI_q^{\alpha} {}_aI_q^{n-\alpha} {}_aD_q^n f(t) \\ &= {}_aI_q^n {}_aD_q^n f(t) \\ &= f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{\Gamma_q(k+1)} {}_aD_q^k f(a). \end{aligned}$$

□

2.3 Impulsive q_k-difference equations

Let $J = [0, T]$, $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$ for $k = 1, 2, \ldots, m$. Let $PC(J, \mathbb{R}) = \{x : J \to \mathbb{R} : x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $k = 1, 2, \ldots, m\}$. $PC(J, \mathbb{R})$ is a Banach space with the norms $\|x\|_{PC} = \sup\{|x(t)|; t \in J\}$.

2.3.1 First-order impulsive q_k-difference equations

In this subsection, we study the existence and uniqueness of solutions for the following initial value problem for first-order impulsive q_k-difference equation

$$\begin{aligned} &D_{q_k} x(t) = f(t, x(t)), \quad t \in J,\ t \neq t_k, \\ &\Delta x(t_k) = I_k\left(x(t_k)\right), \quad k = 1, 2, \ldots, m, \\ &x(0) = x_0, \end{aligned} \tag{2.8}$$

where $x_0 \in \mathbb{R}$, $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_k \in C(\mathbb{R}, \mathbb{R})$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $k = 1, 2, \ldots, m$ and $0 < q_k < 1$ for $k = 0, 1, 2, \ldots, m$.

Lemma 2.3.1 *[3] If $x \in PC(J,\mathbb{R})$ is a solution of (2.8), then for any $t \in J_k$, $k = 0,1,2,\ldots,m$,*

$$x(t) = x_0 + \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} f(s,x(s))d_{q_{k-1}}s + \sum_{0<t_k<t} I_k(x(t_k)) + \int_{t_k}^{t} f(s,x(s))d_{q_k}s, \quad (2.9)$$

with $\sum_{0<0}(\cdot) = 0$, is a solution of (2.8). The converse is also true.

Theorem 2.3.2 *[3] Assume that the following assumptions hold:*
(H_1) $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and satisfies

$$|f(t,x) - f(t,y)| \leq L|x-y|, \quad L > 0, \ \forall t \in J, \ x,y \in \mathbb{R};$$

(H_2) $I_k : \mathbb{R} \to \mathbb{R}$, $k = 1,2,\ldots,m$, are continuous functions and satisfy

$$|I_k(x) - I_k(y)| \leq M|x-y|, \quad M > 0, \ \forall x,y \in \mathbb{R}.$$

If

$$LT + mM \leq \delta < 1,$$

then the nonlinear impulsive q_k-difference initial value problem (2.8) has a unique solution on J.

Example 2.3.3 Consider the following first-order impulsive q_k-difference initial value problem

$$\begin{aligned} D_{\frac{1}{2+k}}x(t) &= \frac{e^{-t}|x(t)|}{(t+\sqrt{5})^2(1+|x(t)|)}, \quad t \in J = [0,1], \ t \neq t_k = \frac{k}{10}, \\ \Delta x(t_k) &= \frac{|x(t_k)|}{12+|x(t_k)|}, \quad k = 1,2,\ldots,9, \\ x(0) &= 0. \end{aligned} \quad (2.10)$$

Here $q_k = 1/(2+k)$, $k = 0,1,2,\ldots,9$, $m = 9$, $T = 1$, $f(t,x) = (e^{-t}|x|)/((t+\sqrt{5})^2(1+|x|))$ and $I_k(x) = |x|/(12+|x|)$. Since $|f(t,x) - f(t,y)| \leq (1/5)|x-y|$ and $|I_k(x) - I_k(y)| \leq (1/12)|x-y|$, then, (H_1), (H_2) are satisfied with $L = (1/5)$, $M = (1/12)$. We can show that

$$LT + mM = \frac{1}{5} + \frac{9}{12} = \frac{19}{20} < 1.$$

Hence, by Theorem 2.3.2, the initial value problem (2.10) has a unique solution on $[0,1]$.

2.3.2 Second-order impulsive q_k-difference equations

In this subsection, we investigate the second-order initial value problem of impulsive q_k-difference equation of the form

$$\begin{aligned} & D_{q_k}^2 x(t) = f(t, x(t)), \quad t \in J,\ t \neq t_k, \\ & \Delta x(t_k) = I_k\left(x(t_k)\right), \quad k = 1, 2, \ldots, m, \\ & D_{q_k} x(t_k^+) - D_{q_{k-1}} x(t_k) = I_k^*\left(x(t_k)\right), \quad k = 1, 2, \ldots, m, \\ & x(0) = \alpha, \quad D_{q_0} x(0) = \beta, \end{aligned} \tag{2.11}$$

where $\alpha, \beta \in \mathbb{R}$, $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_k, I_k^* \in C(\mathbb{R}, \mathbb{R})$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$ for $k = 1, 2, \ldots, m$ and $0 < q_k < 1$ for $k = 0, 1, 2, \ldots, m$.

Lemma 2.3.4 *[3] The unique solution of problem* (2.11) *is given by*

$$\begin{aligned} x(t) = & \alpha + \beta t \\ & + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} (t_k - q_{k-1}s - (1-q_{k-1})t_{k-1}) f(s, x(s)) d_{q_{k-1}} s + I_k\left(x(t_k)\right) \right) \\ & + t \left[\sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} f(s, x(s)) d_{q_{k-1}} s + I_k^*\left(x(t_k)\right) \right) \right] \\ & - \sum_{0<t_k<t} t_k \left(\int_{t_{k-1}}^{t_k} f(s, x(s)) d_{q_{k-1}} s + I_k^*\left(x(t_k)\right) \right) \\ & + \int_{t_k}^{t} (t - q_k s - (1-q_k)t_k) f(s, x(s)) d_{q_k} s, \end{aligned} \tag{2.12}$$

with $\sum_{0<0}(\cdot) = 0$.

Next, we prove the existence and uniqueness of a solution to the initial value problem (2.11). We shall use the Banach's fixed point theorem to accomplish this.

Theorem 2.3.5 *[3] Assume that* (H_1) *and* (H_2) *hold. In addition we suppose that:*

(H_3) $I_k^* : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \ldots, m$, *are continuous functions and satisfy*

$$|I_k^*(x) - I_k^*(y)| \leq M^*|x - y|, \quad M^* > 0,\ \forall x, y \in \mathbb{R}.$$

If

$$\theta := L(\nu_1 + T\nu_2 + \nu_3) + mM + (mT + \nu_4)M^* \leq \delta < 1,$$

where

$$\nu_1 = \sum_{k=1}^{m+1} \frac{(t_k - t_{k-1})^2}{1+q_{k-1}}, \quad \nu_2 = \sum_{k=1}^{m} (t_k - t_{k-1}), \quad \nu_3 = \sum_{k=1}^{m} t_k(t_k - t_{k-1}), \quad \nu_4 = \sum_{k=1}^{m} t_k,$$

then the initial value problem (2.11) *has a unique solution on* J.

Example 2.3.6 Consider the following second-order impulsive q_k-difference initial value problem

$$\begin{aligned} &D^2_{\frac{2}{3+k}} x(t) = \frac{e^{-\sin^2 t}|x(t)|}{(7+t)^2(1+|x(t)|)}, \quad t \in J = [0,1], \ t \neq t_k = \frac{k}{10}, \\ &\Delta x(t_k) = \frac{|x(t_k)|}{5(6+|x(t_k)|)}, \quad k = 1,2,\ldots,9, \\ &D_{\frac{2}{3+k}} x(t_k^+) - D_{\frac{2}{3+k-1}} x(t_k) = \frac{1}{9}\tan^{-1}\left(\frac{1}{5}x(t_k)\right), \quad k = 1,2,\ldots,9, \\ &x(0) = 0, \quad D_{\frac{2}{3}} x(0) = 0. \end{aligned} \tag{2.13}$$

Here $q_k = 2/(3+k)$, $k = 0,1,2,\ldots,9$, $m = 9$, $T = 1$, $f(t,x) = (e^{-\sin^2 t}|x|)/((7+t)^2(1+|x|))$, $I_k(x) = |x|/(5(6+|x|))$ and $I_k^*(x) = (1/9)\tan^{-1}(x/5)$. Since $|f(t,x) - f(t,y)| \le (1/49)|x-y|$, $|I_k(x) - I_k(y)| \le (1/30)|x-y|$ and $|I_k^*(x) - I_k^*(y)| \le (1/45)|x-y|$ then, (H_1), (H_2) and (H_3) are satisfied with $L = (1/49)$, $M = (1/30)$, $M^* = (1/45)$. We find that

$$\nu_1 = \sum_{k=1}^{m+1} \frac{(t_k - t_{k-1})^2}{1+q_{k-1}} = \frac{1380817}{180180}, \qquad \nu_2 = \sum_{k=1}^{m} (t_k - t_{k-1}) = \frac{9}{10},$$
$$\nu_3 = \sum_{k=1}^{m} t_k(t_k - t_{k-1}) = \frac{45}{100}, \qquad \nu_4 = \sum_{k=1}^{m} t_k = \frac{45}{10}.$$

Clearly,

$$L(\nu_1 + T\nu_2 + \nu_3) + mM + (mT + \nu_4)M^* = 0.7839 < 1.$$

Hence, by Theorem 2.3.5, the initial value problem (2.13) has a unique solution on $[0,1]$.

2.4 Fixed Point Theorems

A wide range of problems in nonlinear analysis may be presented in the form of an abstract equation

$$Lu = Nu,$$

where $L : X \to Z$ is a linear operator and $N : Y \to Z$ is a nonlinear operator defined in appropriate normed spaces $X \subset Y$ and Z. For example, the Dirichlet problem fits into this setting, with $Lu := u''$ and $Nu := f(\cdot, u)$. In this case, N is defined, for example, over the set of continuous functions, but L requires twice differentiable functions. Together with the boundary conditions, this yields the following possible choice of X, Y and Z:

$$
\begin{aligned}
X &= \{u \in C^2([0,1]) : u(0) = u(1) = 0\}, \\
Y &= \{u \in C([0,1]) : u(0) = u(1) = 0\}, \\
Z &= C([0,1]).
\end{aligned}
$$

In some cases, one may just consider the restriction of N to X and try to find zeros of the function $F : X \to Z$ given by $Fu = Lu - Nu$; however, in many situations this approach is not enough, and a different analysis is required. In particular, the previous Dirichlet problem is an example of so-called nonresonant problems since the operator $L : X \to Z$ is invertible: for each $\varphi \in Y$, the problem $u''(t) = \varphi(t)$ has a unique solution $u \in X$. Thus, the functional equation $Lu = Nu$ is transformed into a fixed point problem:

$$u = L^{-1}Nu.$$

2.4.1 Contraction Mapping Theorem

Several abstract tools have been developed to deal with problems of this kind; in this chapter, we begin with one of the most popular fixed point theorems in complete metric spaces the contraction mapping theorem. Let X and Y be metric spaces. A mapping $T : X \to Y$ is called a contraction if it is globally Lipschitz with constant $\alpha < 1$. In other wards, $T : X \to Y$ is a contraction if there exists$\alpha < 1$ such that $d(Tx_1, Tx_2) \leq \alpha d(x_1, x_2)$ for all $x_1, x_2 \in X$. Note that used the same d for the distance in both X and Y; this is not a problem, in particular, because we shall only consider the case $X = Y$.

Theorem 2.4.1 *[39]* Let X be a complete metric space, and let $T : X \to X$ be a contraction. Then T has a unique fixed point $\hat{x}$. Furthermore, if x_0 is an arbitrary pint of X and a sequence is defined iteratively by $x_{n+1} = T(x_n)$, then $\hat{x} = \lim_{n\to\infty} x_n$.

The contraction mapping theorem allows a simple and direct proof of the Picard existence and uniqueness theorem. In this case, we want to solve the functional equation

$$x(t) = x_0 + \int_0^t f(s, x(s))ds,$$

so the “obvious ”fixed point operator is

$$Tx(t) := x_0 + \int_0^t f(s, x(s))ds.$$

We only need to find an appropriate complete metric space X such that $T : X \to X$ is well defined and contractive.

To this end, let us first consider constants $\hat{\delta}, r > 0$, such that $K \subset \Omega$, where

$$K := [t_0 - \hat{\delta}, t_0 + \hat{\delta}] \times \bar{B}_r(x_0).$$

Next, define $M := \|f_{|K|}\|_\infty$ and L as the Lipschitz constant of f over K, and let

$$X := \{x \in C[t_0 - \delta, t_0 + \delta], \mathbb{R}^n) : x(t) \in \bar{B}_r(x_0) \text{ for all } t.$$

for some $\delta \leq \bar{\delta}$ to be established. In other words, X is just the closed ball of radius r centered in x_0 in the space $C([t_0 - \delta, t_0 + \delta], \mathbb{R}^n)$, equipped with the usual metric

$$d(x, y) = \max_{t \in [t_0 - \delta, t_0 + \delta]} |x(t) - y(t)|.$$

It is clear that $T : X \to C([t_0 - \delta, t_0 + \delta], \mathbb{R}^\times)$ is well defined and, moreover,

$$|Tx(t) - x_0| = \left| \int_{t_0}^t f(s, x(s))ds \right| \leq M\delta.$$

Choosing $\delta \leq \frac{r}{M}$, it follows that X is an invariant set, i.e. $T(X) \subset X$. On the other hand for $x, y \in X$, then

$$d(Tx, Ty) = \max_{t \in [t_0 - \delta, t_0 + \delta]} \left| \int_{t_0}^t (f(s, x(s)) - f(s, y(s)))ds \right| \leq \delta L d(x, y)$$

Hence, it suffices to take $\delta < \min\left\{\frac{r}{M}, \frac{1}{L}\right\}$, and then $T : X \to X$ is a contraction.

Although the Banach theorem ensures that the fixed point is unique, an extra step is needed for the uniqueness invoked in Theorem 2.4.1 since, in principle, for the same δ there might be other solutions that abandon the ball $\bar{B}_r(x_0)$. One possible line of reasoning is as follows: suppose y is another solution and fix $\bar{\delta} \in (0, \delta]$ such that $|y(t)| \leq r$ for $t \in [t_0 - \bar{\delta}, t_0 + \bar{\delta}]$. Next, redefine the space X accordingly, so the operator T has a unique fixed point and thus $x = y$ on $[t_0 - \bar{\delta}, t_0 + \bar{\delta}]$. This proves only local uniqueness, in the sense that two solutions must coincide in a neighborhood of t_0. But now the same existence and (local)uniqueness result does the rest of the job: suppose two solutions x and y are defined over an open interval I containing t_0; then the set $J := \{t \in I : x(t) = y(t)\}$ is closed in I and nonempty. Moreover, if $t_1 \in J$, then $x(t_1) = y(t_1)$, and hence $x = y$ in a neighborhood of t_1. This shows that J is open and, consequently, $J = I$.

Theorem 2.4.2 *[39]* Let X be a complete metric spaces, and let $T : X \to X$ be a mapping. If $T^n := T \circ T \circ \cdots T$ (n times) is a contraction for some positive integer n, then T has a unique fixed point.

Theorem 2.4.3 *[39]* Let $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and globally Lipschitz with respect to x with constant L. Then for any $(t_0, x_0) \in (a, b) \times \mathbb{R}^n$ the unique solution of the problem

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

is defined over $[a, b]$.

The contraction mapping theorem is an efficient tool for proving existence and uniqueness, although its application might also be quite restrictive. The assumption that f is globally Lipschitz is already strong; furthermore, we have required the Lipschitz constant to be small. Nevertheless, there are many cases in which this assumption can be relaxed. In Picard's fundamental theorem, this was easy: only a local Lipschitz assumption was required since we were looking for local solutions; in other situations, the global Lipschitz condition may be avoided if one is able to obtain a priori bounds for the solutions, as we saw in the first chapter. But, still, one must prove that the fixed point operator is contractive: this explains why the Lipschitz constant must be small. The smallness assumption can be dropped when the operator has some other properties, such as monotonicity.

Theorem 2.4.4 *[39]* Let H be a Hilbert space, and assume that $T : H \to H$ is globally Lipschitz and monotone nonincreasing, that is

$$\langle Tx - Ty, x - y \rangle \leq 0$$

for all $x, y \in H$. Then for each fixed $y \in H$ the equation $x = Tx + y$ has a unique solution. In particular, T has a unique fixed point.

2.4.2 Kranoselskii Fixed Point Theorem

Theorem 2.4.5 *[40]* Let K be a non-empty complete convex subset of a normed space E, let A be a continuous mapping of K into a compact subset of E, let B map K and satisfy a Lipschitz condition

$$\|Bx - Bx'\| \leq k\|x - x'\| \quad (x, x' \in K),$$

with $0 < k < 1$ and let $Ax + By \in K$ for all $x, y \in K$. Then there is a point $u \in K$ with

$$Au + Bu = u.$$

Corollary 2.4.6 *[40]* Let K be a non-empty complete convex subset of a normed space, let A be a continuous of K into a compact subset of K, let B map K into itself and satisfy the Lipschitz condition

$$\|Bx - Bx'\| \leq \|x - x'\| \quad (x, x' \in K),$$

and let $0 < \alpha < 1$. Then there exists a point $u \in K$ with

$$\alpha Au + (1 - \alpha)Bu = u.$$

In general, under the condition of Schauder's theorem, we have no method for the calculation of a fixed point of a mapping. However there is a special case in which this can be done using a method due to Kranoselskii.

Definition 2.4.7 A norm p is uniformly convex if it satisfies

$$p(x_n) = p(y_n) = 1 \quad (n = 1, 2, \ldots), \quad \lim_{n\to\infty} p(x_n + y_n) = 2 \Rightarrow \lim_{n\to\infty} p(x_n - y_n) = 0.$$

Lemma 2.4.8 *[40]* Let p be a uniformly convex norm, and let εM be positive constants. Then there exists a constant δ with $0 < \delta < 1$ such that

$$p(x) \leq M, \quad p(y) \leq M, \quad p(x - y) \geq \varepsilon \Rightarrow p(x + y) \leq 2\delta \max(p(x), p(y)).$$

Theorem 2.4.9 *[40]* Let K be a bounded closed convex set in a Banach space E with a uniformly convex norm. Let T be a mapping of K into a compact subset of K that satisfies a Lipschitz condition with Lipschitz constant 1, and let x_0 be an arbitrary point of K. Then the sequence defined by

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n) \quad (n = 0, 1, 2, \ldots)$$

converges to a fixed point of T in K.

2.4.3 The Leray-Schauder Fixed point Theorem

We apply the topological transversality theorem to the equation $x = F(x)$, where F is a compact or completely continuous operator.

Theorem 2.4.10 *[41]* (Leray-Shauder principle). Let $C \subset E$ be a convex set, and let U be open in C. Let $\{H_t : \bar{U} \to C\}$ be an admissible compact homotopy such that $H_0 = F$ and $H_1 = G$, where G is the constant map sending $\bar{U}$ to a point $u_0 \in U$. Then F has a fixed point.

Theorem 2.4.11 *[41]* (Nonlinear alternative). Let $C \subset E$ be a convex set, and let U be open in C and such that $0 \in U$. Then each compact map $F : \bar{U} \to C$ has at least one of the following two properties:

(a) F has a fixed point,

(b) there exist $x \in \partial U$ and $\lambda \in (0,1)$ such that $x = \lambda F(x)$.

Many of the customary fixed point theorems can be derived from the nonlinear alternative by imposing conditions that prevent occurrence of the second property. As an illustration of such conditions, let $p : E \to \mathbb{R}^+$ be any (not necessarily continuous) function such that $p^{-1}(0) = 0$ and $p(\lambda x) = \lambda p(x)$ for all $\lambda > 0$; any norm, not necessarily equivalent to the given one in E, is an example of such a function. Then we have

Corollary 2.4.12 *[41]* Let $C \subset E$ be convex, and $U \subset C$ an open subset that contain 0. Let $F : \bar{U} \to C$ be a compact map. If either

- Rothe type condition: $p[F(x)] \leq p(x)$ for all $x \in \partial U$,
- Altman type condition: $[pF(x)]^2 \leq [p(F(x) - x)]^2 + [p(x)]^2$ for all $x \in \partial U$,

then F has a fixed point.

Theorem 2.4.13 *[41]* (Leray-Schauder alternative). Let C be a convex subset of E, and assume $0 \in C$. Let $F : C \to C$ be a completely continuous operator, and let

$$\varepsilon(F) = \{x \in C | x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

Then either $\epsilon(F)$ is unbounded or F has a fixed point.

Theorem 2.4.14 *[41]* Let $F : E \times I \to E$ be a completely continuous operator such that for some $\delta > 0$, $F(x,t) = -F(-x,0)$ for all $x \in E$ with $\|x\| \geq \delta$. Let

$$\varepsilon(F) = \{x \in E | x = F(x,t) \text{ for some } t \in (0,1)\}.$$

Then either $\varepsilon(F)$ is unbounded or $x \mapsto F(x,1)$ has a fixed point.

Consider the compact homotopy $F_t(x) = F(x,t)$ for $(x,t) \in \bar{B} \times I$; we may assume that F_t is fixed point free on δB, so that $F_0 \cong F_1$ in $\mathcal{H}_{\partial B}(\bar{B}, E)$. Then since F_0 is antipode-preserving on ∂B, the desired conclusion follows.

Recall that an operator $F : E \to E$ is called quasi-bounded whenever

$$\|F\| = \overline{\lim}_{\|x\| \to \infty} \frac{\|F(x)\|}{\|x\|} = \inf_{\delta > 0} \sup_{\|x\| \geq \delta} \frac{\|F(x)\|}{\|x\|} < \infty.$$

Theorem 2.4.15 *[41]* Let $F : E \to E$ be a quasi-bonded completely continuous operator. Then for each real $|\lambda| < 1/\|F\|$ (and for all real λ whenever $\|F\| = 0$) the operator λF has at least one fixed point. More generally: for each $y \in E$ and $|\lambda| < 1/\|F\|$ the equation

$$y = x - \lambda F(x)$$

has at least one solution.

2.4.4 Multi-Value Mapping

Now, we recall some definitions and notations about multifunctions ([42], [43]).

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed }\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(\mathcal{X}) : \mathcal{Y} \text{ is bounded }\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact }\}$, and $\mathcal{P}_{cp,cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex }\}$.

A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = U_{x\in\mathbb{B}}G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$.) G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : J \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable.

We define the graph of a function G to be the set $Gr(G) = \{(x, y) \in X \times Y, y \in G(x)\}$ and recall two results for closed graphs and upper semi-continuity.

Lemma 2.4.16 *([42], Proposition 1.2) Let $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty, x_n \to x_*, y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

Now we state some known fixed point theorem which is needed in the sequel.

Lemma 2.4.17 *(Nonlinear alternative for Kakutani maps)[44]. Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp,cv}(C)$ is a upper semi-continuous compact map. Then either*

(i) F has a fixed point in $\overline{U}$, or

(ii) there is a $u \in \partial U$ and $\lambda \in (0, 1)$ with $u \in \lambda F(u)$.

Lemma 2.4.18 *([45]) Let X be a Banach space. Let $F : J \times \mathbb{R}^3 \to \mathcal{P}_{cp,cv}(X)$ be an L^1- Carathéodory function and let Θ be a linear continuous mapping from $L^1(J, \mathbb{R})$ to $C(J, \mathbb{R})$. Then the operator*

$$\Theta \circ S_F : C(J, \mathbb{R}) \to \mathcal{P}_{cp,cv}(C(J, \mathbb{R})), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x}),$$

is a closed graph operator in $C(J, \mathbb{R}) \times C(J, \mathbb{R})$.

Lemma 2.4.19 *([47]) Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1(J, \mathbb{R}))$ be a multi-valued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1(J, \mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Lemma 2.4.20 *([48]) Let (X, d) be a complete metric space. If $N : X \to \mathcal{P}_{]\updownarrow}(\mathcal{X})$ is a contraction, then $FixN \neq \emptyset$.*

Chapter 3

Research Methodology

In this chapter we transform the boundary value problem (1.4) and (1.5) into integral equations.

3.1 Integral Equation of (1.4)

For the sake of convenience, we introduce the following notations which will be used to compute some quantum constants. For nonnegative integers $a < b$, we have

$$\Omega(a,b) = \prod_{j=a}^{b-1} \frac{(t_{j+1}-t_j)^{1-\gamma_j}}{\Gamma_{q_j}(2-\gamma_j)}, \tag{3.1}$$

$$\Psi(a,b) = \sum_{i=a}^{b-1} (t_{i+1}-t_i)\Omega(a,i), \tag{3.2}$$

with $\prod_c^d(\cdot) = 1$, $\sum_c^d(\cdot) = 0$, if $c > d$. For example,

$$\Psi(2,5) = (t_3-t_2) + (t_4-t_3)\frac{(t_3-t_2)^{1-\gamma_2}}{\Gamma_{q_2}(2-\gamma_2)} + (t_5-t_4)\frac{(t_4-t_3)^{1-\gamma_3}}{\Gamma_{q_3}(2-\gamma_3)}\frac{(t_3-t_2)^{1-\gamma_2}}{\Gamma_{q_2}(2-\gamma_2)}.$$

The following formulas expressed in terms of above notations will be used in the sequel.

Property 3.1.1 *Let $a < b$ be nonnegative integers. The following relations hold:*

(P_1) $\Psi(a,b) + (t_{b+1}-t_b)\Omega(a,b) = \Psi(a,b+1)$,

(P_2) $\displaystyle\sum_{i=a}^{b-1} \Psi(i,b) + (t_{b+1}-t_b)\sum_{i=a}^{b} \Omega(i,b) = \sum_{i=a}^{b} \Psi(i,b+1)$.

Now we present an auxiliary lemma which plays a pivotal role in the forthcoming analysis.

Lemma 3.1.2 *Let* $\lambda_1(\xi_1\Psi(0,m+1)+\xi_2\Omega(0,m+1)) \neq \xi_1\lambda_2$ *and* $g \in C(J,\mathbb{R})$. *Then the unique solution of the linear problem*

$$\begin{cases} {}^{c}_{t_k}D^{\alpha_k}_{q_k}x(t) = g(t), \quad t \in J_k \subseteq [0,T],\ t \neq t_k, \\ \Delta x(t_k) = x(t_k^+) - x(t_k) = \varphi_k\left(x(t_k)\right), \quad k = 1,2,\ldots,m, \\ {}_{t_k}D_{q_k}x(t_k^+) - {}^{c}_{t_{k-1}}D^{\gamma_{k-1}}_{q_{k-1}}x(t_k) = \varphi_k^*\left(x(t_k)\right), \quad k = 1,2,\ldots,m, \\ \lambda_1 x(0) + \lambda_2\, {}_0D_{q_0}x(0) = 0, \qquad \xi_1 x(T) + \xi_2\, {}^{c}_{t_m}D^{\gamma_m}_{q_m}x(T) = 0, \end{cases} \tag{3.3}$$

is given by

$$\begin{aligned} x(t) \;=\; & \frac{1}{\Delta}\{\lambda_2 - \lambda_1(\Psi(0,k) + (t-t_k)\Omega(0,k))\}\Bigg(\xi_1\Bigg\{\sum_{i=0}^{m-1}\left[{}_{t_i}I^{\alpha_i}_{q_i}g(t_{i+1}) + \varphi_{i+1}\left(x(t_{i+1})\right)\right] \\ & + \sum_{i=0}^{m-1}\Psi(i+1,m+1)\left[{}_{t_i}I^{\alpha_i-\gamma_i}_{q_i}g(t_{i+1}) + \varphi^*_{i+1}\left(x(t_{i+1})\right)\right] + {}_{t_m}I^{\alpha_m}_{q_m}g(T)\Bigg\} \\ & + \xi_2\Bigg\{\sum_{i=0}^{m-1}\Omega(i+1,m+1)\left[{}_{t_i}I^{\alpha_i-\gamma_i}_{q_i}g(t_{i+1}) + \varphi^*_{i+1}\left(x(t_{i+1})\right)\right] + {}_{t_m}I^{\alpha_m-\gamma_m}_{q_m}g(T)\Bigg\}\Bigg) \\ & + \sum_{i=0}^{k-1}\left[{}_{t_i}I^{\alpha_i}_{q_i}g(t_{i+1}) + \varphi_{i+1}\left(x(t_{i+1})\right)\right] + \sum_{i=0}^{k-2}\Psi(i+1,k)\left[{}_{t_i}I^{\alpha_i-\gamma_i}_{q_i}g(t_{i+1}) + \varphi^*_{i+1}\left(x(t_{i+1})\right)\right] \\ & + (t-t_k)\sum_{i=0}^{k-1}\Omega(i+1,k)\left[{}_{t_i}I^{\alpha_i-\gamma_i}_{q_i}g(t_{i+1}) + \varphi^*_{i+1}\left(x(t_{i+1})\right)\right] + {}_{t_k}I^{\alpha_k}_{q_k}g(t). \end{aligned} \tag{3.4}$$

where $\Delta = \lambda_1(\xi_1\Psi(0,m+1) + \xi_2\Omega(0,m+1)) - \xi_1\lambda_2$.

Proof. Using the Riemann-Liouville fractional q_0-integral of order α_0 to both sides of the first equation of (3.3) for $t \in J_0$ and applying Lemma 2.2.7, we have

$${}_{t_0}I^{\alpha_0}_{q_0}\,{}^{c}_{t_0}D^{\alpha_0}_{q_0}x(t) = x(t) - x(0) - \frac{{}_0D_{q_0}x(0)}{\Gamma_{q_0}(2)}t = {}_{t_0}I^{\alpha_0}_{q_0}g(t),$$

which leads to

$$x(t) = C_0 + C_1 t + {}_{t_0}I^{\alpha_0}_{q_0}g(t), \tag{3.5}$$

where $C_0 = x(0)$ and $C_1 = {}_0D_{q_0}x(0)$. From the definition 2.2.6 and (2.6), we get

$${}^{c}_{t_k}D^{\gamma_k}_{q_k}C = 0, \qquad {}^{c}_{t_k}D^{\gamma_k}_{q_k}(t-t_k) = \frac{(t-t_k)^{1-\gamma_k}}{\Gamma_{q_k}(2-\gamma_k)}, \quad k = 0,1,2,\ldots,m,\ C \in \mathbb{R}.$$

Then we get for $t = t_1$ that

$$x(t_1) = C_0 + C_1 t_1 + {}_{t_0}I_{q_0}^{\alpha_0} g(t_1) \quad \text{and} \quad {}_{t_0}^{c}D_{q_0}^{\gamma_0} x(t_1) = C_1 \frac{(t_1 - t_0)^{1-\gamma_0}}{\Gamma_{q_0}(2-\gamma_0)} + {}_{t_0}I_{q_0}^{\alpha_0-\gamma_0} g(t_1). \tag{3.6}$$

For $t \in J_1$, again taking the Riemann-Liouville fractional q_1-integral of order α_1 to (3.3) and using the above process, we get

$$x(t) = x(t_1^+) + (t - t_1)_{t_1}D_{q_1} x(t_1^+) + {}_{t_1}I_{q_1}^{\alpha_1} g(t). \tag{3.7}$$

Applying the impulsive conditions $x(t_1^+) = x(t_1) + \varphi_1(x(t_1))$ and ${}_{t_1}D_{q_1} x(t_1^+) = {}_{t_0}^{c}D_{q_0}^{\gamma_0} x(t_1) + \varphi_1^*(x(t_1))$, we get

$$\begin{aligned} x(t) &= C_0 + C_1 \left[t_1 + (t - t_1) \frac{(t_1 - t_0)^{1-\gamma_0}}{\Gamma_{q_0}(2-\gamma_0)} \right] + \left[{}_{t_0}I_{q_0}^{\alpha_0} g(t_1) + \varphi_1(x(t_1)) \right] \\ &\quad + (t - t_1) \left[{}_{t_0}I_{q_0}^{\alpha_0-\gamma_0} g(t_1) + \varphi_1^*(x(t_1)) \right] + {}_{t_1}I_{q_1}^{\alpha_1} g(t). \end{aligned}$$

In the same ways, for $t \in J_2$, we have

$$\begin{aligned} x(t) &= C_0 + C_1 \left\{ (t_1 - t_0) + (t_2 - t_1) \frac{(t_1 - t_0)^{1-\gamma_0}}{\Gamma_{q_0}(2-\gamma_0)} + (t - t_2) \frac{(t_2 - t_1)^{1-\gamma_1}}{\Gamma_{q_1}(2-\gamma_1)} \frac{(t_1 - t_0)^{1-\gamma_0}}{\Gamma_{q_0}(2-\gamma_0)} \right\} \\ &\quad + \left[{}_{t_0}I_{q_0}^{\alpha_0} g(t_1) + \varphi_1(x(t_1)) \right] + \left[{}_{t_1}I_{q_1}^{\alpha_1} g(t_2) + \varphi_2(x(t_2)) \right] \\ &\quad + (t - t_2) \left\{ \frac{(t_2 - t_1)^{1-\gamma_1}}{\Gamma_{q_1}(2-\gamma_1)} \left[{}_{t_0}I_{q_0}^{\alpha_0-\gamma_0} g(t_1) + \varphi_1^*(x(t_1)) \right] \right. \\ &\quad \left. + \left[{}_{t_1}I_{q_1}^{\alpha_1-\gamma_1} g(t_2) + \varphi_2^*(x(t_2)) \right] \right\} + (t_2 - t_1) \left[{}_{t_0}I_{q_0}^{\alpha_0-\gamma_0} g(t_1) + \varphi_1^*(x(t_1)) \right] \\ &\quad + {}_{t_2}I_{q_2}^{\alpha_2} g(t). \end{aligned}$$

Repeating the above process and taking into account of (3.1)-(3.2), for $t \in J_k \subseteq J$, $k = 0, 1, 2, \ldots, m$, we have

$$\begin{aligned} x(t) &= C_0 + C_1 \left\{ \Psi(0, k) + (t - t_k)\Omega(0, k) \right\} + \sum_{i=0}^{k-1} \left[{}_{t_i}I_{q_i}^{\alpha_i} g(t_{i+1}) + \varphi_{i+1}(x(t_{i+1})) \right] \\ &\quad + \sum_{i=0}^{k-2} \Psi(i+1, k) \left[{}_{t_i}I_{q_i}^{\alpha_i-\gamma_i} g(t_{i+1}) + \varphi_{i+1}^*(x(t_{i+1})) \right] \\ &\quad + (t - t_k) \sum_{i=0}^{k-1} \Omega(i+1, k) \left[{}_{t_i}I_{q_i}^{\alpha_i-\gamma_i} g(t_{i+1}) + \varphi_{i+1}^*(x(t_{i+1})) \right] + {}_{t_k}I_{q_k}^{\alpha_k} g(t). \end{aligned} \tag{3.8}$$

From (3.14), we find that

$$\begin{aligned} x(T) &= C_0 + C_1\left\{\Psi(0,m) + (T-t_m)\Omega(0,m)\right\} + \sum_{i=0}^{m-1}\left[{}_{t_i}I_{q_i}^{\alpha_i} g(t_{i+1}) + \varphi_{i+1}\left(x(t_{i+1})\right)\right] \\ &\quad + \sum_{i=0}^{m-2}\Psi(i+1,m)\left[{}_{t_i}I_{q_i}^{\alpha_i-\gamma_i} g(t_{i+1}) + \varphi_{i+1}^*\left(x(t_{i+1})\right)\right] \\ &\quad + (T-t_m)\sum_{i=0}^{m-1}\Omega(i+1,m)\left[{}_{t_i}I_{q_i}^{\alpha_i-\gamma_i} g(t_{i+1}) + \varphi_{i+1}^*\left(x(t_{i+1})\right)\right] + {}_{t_m}I_{q_m}^{\alpha_m} g(T) \\ &= C_0 + C_1\Psi(0,m+1) + \sum_{i=0}^{m-1}\left[{}_{t_i}I_{q_i}^{\alpha_i} g(t_{i+1}) + \varphi_{i+1}\left(x(t_{i+1})\right)\right] \\ &\quad + \sum_{i=0}^{m-1}\Psi(i+1,m+1)\left[{}_{t_i}I_{q_i}^{\alpha_i-\gamma_i} g(t_{i+1}) + \varphi_{i+1}^*\left(x(t_{i+1})\right)\right] + {}_{t_m}I_{q_m}^{\alpha_m} g(T), \end{aligned}$$

and

$$\begin{aligned} {}_{t_m}^{c}D_{q_m}^{\gamma_m} x(T) &= C_1\Omega(0,m+1) + \sum_{i=0}^{m-1}\Omega(i+1,m+1)\left[{}_{t_i}I_{q_i}^{\alpha_i-\gamma_i} g(t_{i+1}) + \varphi_{i+1}^*\left(x(t_{i+1})\right)\right] \\ &\quad + {}_{t_m}I_{q_m}^{\alpha_m-\gamma_m} g(T). \end{aligned}$$

From the boundary condition of (3.3), we find that

$$\begin{aligned} C_0 &= \frac{\lambda_2\xi_1}{\Delta}\left\{\sum_{i=0}^{m-1}\left[{}_{t_i}I_{q_i}^{\alpha_i} g(t_{i+1}) + \varphi_{i+1}\left(x(t_{i+1})\right)\right]\right. \\ &\quad \left. + \sum_{i=0}^{m-1}\Psi(i+1,m+1)\left[{}_{t_i}I_{q_i}^{\alpha_i-\gamma_i} g(t_{i+1}) + \varphi_{i+1}^*\left(x(t_{i+1})\right)\right] + {}_{t_m}I_{q_m}^{\alpha_m} g(T)\right\} \\ &\quad + \frac{\lambda_2\xi_2}{\Delta}\left\{\sum_{i=0}^{m-1}\Omega(i+1,m+1)\left[{}_{t_i}I_{q_i}^{\alpha_i-\gamma_i} g(t_{i+1}) + \varphi_{i+1}^*\left(x(t_{i+1})\right)\right] + {}_{t_m}I_{q_m}^{\alpha_m-\gamma_m} g(T)\right\} \end{aligned}$$

and

$$\begin{aligned} C_1 &= -\frac{\lambda_1\xi_1}{\Delta}\left\{\sum_{i=0}^{m-1}\left[{}_{t_i}I_{q_i}^{\alpha_i} g(t_{i+1}) + \varphi_{i+1}\left(x(t_{i+1})\right)\right]\right. \\ &\quad \left. + \sum_{i=0}^{m-1}\Psi(i+1,m+1)\left[{}_{t_i}I_{q_i}^{\alpha_i-\gamma_i} g(t_{i+1}) + \varphi_{i+1}^*\left(x(t_{i+1})\right)\right] + {}_{t_m}I_{q_m}^{\alpha_m} g(T)\right\} \\ &\quad - \frac{\lambda_1\xi_2}{\Delta}\left\{\sum_{i=0}^{m-1}\Omega(i+1,m+1)\left[{}_{t_i}I_{q_i}^{\alpha_i-\gamma_i} g(t_{i+1}) + \varphi_{i+1}^*\left(x(t_{i+1})\right)\right] + {}_{t_m}I_{q_m}^{\alpha_m-\gamma_m} g(T)\right\}. \end{aligned}$$

Substituting the values C_0 and C_1 in (3.14), we obtain the unique solution (3.4). This completes the proof. □

3.2 Integral Equation of (1.5)

Lemma 3.2.1 *Let $h \in C(J, \mathbb{R})$. Then the unique solution of*

$$\begin{cases} {}^{c}_{t_k}D^{\alpha_k}_{q_k}x(t) = h(t), \quad t \in J_k \subseteq [0,T], \ t \neq t_k, \\ \Delta x(t_k) = \varphi_k\left({}_{t_{k-1}}I^{\beta_{k-1}}_{q_{k-1}}x(t_k)\right), \quad k = 1,2,\ldots,m, \\ {}_{t_k}D_{q_k}x(t_k^+) - {}_{t_{k-1}}D_{q_{k-1}}x(t_k) = \varphi_k^*\left({}_{t_{k-1}}I^{\gamma_{k-1}}_{q_{k-1}}x(t_k)\right), \quad k = 1,2,\ldots,m, \\ x(0) = -x(T), \qquad {}_0D_{q_0}x(0) = -{}_{t_m}D_{q_m}x(T), \end{cases} \tag{3.9}$$

is given by

$$\begin{aligned} x(t) \ = \ & -\frac{1}{2}\sum_{i=1}^{m}\left[{}_{t_{i-1}}I^{\alpha_{i-1}}_{q_{i-1}}h(t_i) + \varphi_i\left({}_{t_{i-1}}I^{\beta_{i-1}}_{q_{i-1}}x(t_i)\right)\right] \\ & -\frac{1}{2}\sum_{i=1}^{m}(T-t_i)\left\{{}_{t_{i-1}}I^{\alpha_{i-1}-1}_{q_{i-1}}h(t_i) + \varphi_i^*\left({}_{t_{i-1}}I^{\gamma_{i-1}}_{q_{i-1}}x(t_i)\right)\right\} \\ & -\frac{1}{2}{}_{t_m}I^{\alpha_m}_{q_m}h(T) + \left(t - \frac{T}{2}\right)\left[-\frac{1}{2}\sum_{i=1}^{m}\left\{{}_{t_{i-1}}I^{\alpha_{i-1}-1}_{q_{i-1}}h(t_i)\right.\right. \\ & \left.\left. +\varphi_i^*\left({}_{t_{i-1}}I^{\gamma_{i-1}}_{q_{i-1}}x(t_i)\right)\right\} - \frac{1}{2}{}_{t_m}I^{\alpha_m-1}_{q_m}h(T)\right] \\ & +\sum_{i=1}^{k}\left[{}_{t_{i-1}}I^{\alpha_{i-1}}_{q_{i-1}}h(t_i) + \varphi_i\left({}_{t_{i-1}}I^{\beta_{i-1}}_{q_{i-1}}x(t_i)\right)\right] \\ & +\sum_{i=1}^{k}(t-t_i)\left\{{}_{t_{i-1}}I^{\alpha_{i-1}-1}_{q_{i-1}}h(t_i) + \varphi_i^*\left({}_{t_{i-1}}I^{\gamma_{i-1}}_{q_{i-1}}x(t_i)\right)\right\} \\ & +{}_{t_k}I^{\alpha_k}_{q_k}h(t), \end{aligned} \tag{3.10}$$

where $\sum_1^0(\cdot) = 0$.

Proof. Applying the Riemann-Liouville fractional q_0-integral operator of order α_0 on both sides of the first equation of (3.9) for $t \in J_0$ and using Lemma 2.2.7, we obtain

$${}_{t_0}I^{\alpha_0}_{q_0}{}^{c}_{t_0}D^{\alpha_0}_{q_0}x(t) = x(t) - x(0) - \frac{{}_0D_{q_0}x(0)}{\Gamma_{q_0}(2)}t = {}_{t_0}I^{\alpha_0}_{q_0}h(t),$$

which yields

$$x(t) = C_0 + C_1 t + {}_{t_0}I^{\alpha_0}_{q_0}h(t), \tag{3.11}$$

where $C_0 = x(0)$ and $C_1 = {}_0D_{q_0}x(0)$. In particular, for $t = t_1$, we have

$$x(t_1) = C_0 + C_1 t_1 + {}_{t_0}I^{\alpha_0}_{q_0}h(t_1) \quad \text{and} \quad {}_{t_0}D_{q_0}x(t_1) = C_1 + {}_{t_0}I^{\alpha_0-1}_{q_0}h(t_1). \tag{3.12}$$

For $t \in J_1$, application of the Riemann-Liouville fractional q_1-integral operator of order α_1 to (3.9) and using the above arguments, we get

$$x(t) = x(t_1^+) + (t - t_1)_{t_1}D_{q_1}x(t_1^+) + {}_{t_1}I_{q_1}^{\alpha_1}h(t). \tag{3.13}$$

Using the impulsive conditions $x(t_1^+) = x(t_1) + \varphi_1\left({}_{t_0}I_{q_0}^{\beta_0}x(t_1)\right)$ and ${}_{t_1}D_{q_1}x(t_1^+) = {}_{t_0}D_{q_0}x(t_1) + \varphi_1^*\left({}_{t_0}I_{q_0}^{\gamma_0}x(t_1)\right)$, we obtain

$$\begin{aligned} x(t) &= C_0 + C_1 t + \left[{}_{t_0}I_{q_0}^{\alpha_0}h(t_1) + \varphi_1\left({}_{t_0}I_{q_0}^{\beta_0}x(t_1)\right)\right] \\ &\quad + (t - t_1)\left[{}_{t_0}I_{q_0}^{\alpha_0-1}h(t_1) + \varphi_1^*\left({}_{t_0}I_{q_0}^{\gamma_0}x(t_1)\right)\right] + {}_{t_1}I_{q_1}^{\alpha_1}h(t). \end{aligned}$$

In a similar manner, for $t \in J_2$, we have

$$\begin{aligned} x(t) &= C_0 + C_1 t + \left[{}_{t_0}I_{q_0}^{\alpha_0}h(t_1) + \varphi_1\left({}_{t_0}I_{q_0}^{\beta_0}x(t_1)\right)\right] + \left[{}_{t_1}I_{q_1}^{\alpha_1}h(t_2) + \varphi_2\left({}_{t_1}I_{q_1}^{\beta_1}x(t_2)\right)\right] \\ &\quad + (t - t_1)\left[{}_{t_0}I_{q_0}^{\alpha_0-1}h(t_1) + \varphi_1^*\left({}_{t_0}I_{q_0}^{\gamma_0}x(t_1)\right)\right] \\ &\quad + (t - t_2)\left[{}_{t_1}I_{q_1}^{\alpha_1-1}h(t_2) + \varphi_2^*\left({}_{t_1}I_{q_1}^{\gamma_1}x(t_2)\right)\right] + {}_{t_2}I_{q_2}^{\alpha_2}h(t). \end{aligned}$$

Repeating the above process, for $t \in J_k \subseteq J$, $k = 0, 1, 2, \ldots, m$, we obtain

$$\begin{aligned} x(t) &= C_0 + C_1 t + \sum_{i=1}^{k}\left[{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}}h(t_i) + \varphi_i\left({}_{t_{i-1}}I_{q_{i-1}}^{\beta_{i-1}}x(t_i)\right)\right] \\ &\quad + \sum_{i=1}^{k}(t - t_i)\left\{{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1}h(t_i) + \varphi_i^*\left({}_{t_{i-1}}I_{q_{i-1}}^{\gamma_{i-1}}x(t_i)\right)\right\} \\ &\quad + {}_{t_k}I_{q_k}^{\alpha_k}h(t), \end{aligned} \tag{3.14}$$

where $\sum_1^0(\cdot) = 0$. Notice that $x(0) = C_0$ and

$$\begin{aligned} x(T) &= C_0 + C_1 T + \sum_{i=1}^{m}\left[{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}}h(t_i) + \varphi_i\left({}_{t_{i-1}}I_{q_{i-1}}^{\beta_{i-1}}x(t_i)\right)\right] \\ &\quad + \sum_{i=1}^{m}(T - t_i)\left\{{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1}h(t_i) + \varphi_i^*\left({}_{t_{i-1}}I_{q_{i-1}}^{\gamma_{i-1}}x(t_i)\right)\right\} \\ &\quad + {}_{t_m}I_{q_m}^{\alpha_m}h(T). \end{aligned}$$

On the other hand, we have

$${}_{t_k}D_{q_k}x(t) = C_1 + \sum_{i=1}^{k}\left\{{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1}h(t_i) + \varphi_i^*\left({}_{t_{i-1}}I_{q_{i-1}}^{\gamma_{i-1}}x(t_i)\right)\right\} + {}_{t_k}I_{q_k}^{\alpha_k-1}h(t),$$

which implies ${}_{t_0}D_{q_0}x(0) = C_1$ and

$${}_{t_m}D_{q_m}x(T) = C_1 + \sum_{i=1}^{m}\left\{{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1}h(t_i) + \varphi_i^*\left({}_{t_{i-1}}I_{q_{i-1}}^{\gamma_{i-1}}x(t_i)\right)\right\} + {}_{t_m}I_{q_m}^{\alpha_m-1}h(T).$$

Now making use of the boundary conditions given by (3.9), we find that

$$C_0 = -\frac{1}{2}C_1T - \frac{1}{2}\sum_{i=1}^{m}\left[{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}}h(t_i) + \varphi_i\left({}_{t_{i-1}}I_{q_{i-1}}^{\beta_{i-1}}x(t_i)\right)\right]$$
$$-\frac{1}{2}\sum_{i=1}^{m}(T-t_i)\left\{{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1}h(t_i) + \varphi_i^*\left({}_{t_{i-1}}I_{q_{i-1}}^{\gamma_{i-1}}x(t_i)\right)\right\} - \frac{1}{2}{}_{t_m}I_{q_m}^{\alpha_m}h(T),$$

and

$$C_1 = -\frac{1}{2}\sum_{i=1}^{m}\left\{{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1}h(t_i) + \varphi_i^*\left({}_{t_{i-1}}I_{q_{i-1}}^{\gamma_{i-1}}x(t_i)\right)\right\} - \frac{1}{2}{}_{t_m}I_{q_m}^{\alpha_m-1}h(T).$$

Substituting the values C_0 and C_1 in (3.14) yields the solution (3.10). □

Chapter 4

Main Results

Let $PC(J,\mathbb{R}) = \{x : J \to \mathbb{R} : x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $k = 1, 2, \ldots, m\}$. Observe that $PC(J,\mathbb{R})$ is a Banach space equipped with the norm $\|x\|_{PC} = \sup\{|x(t)| : t \in J\}$.

In view of Lemma 3.1.2, we define an operator $\mathcal{Q} : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ by

$$
\begin{aligned}
&\mathcal{Q}x(t)\\
=\ & \frac{1}{\Delta}\left\{\lambda_2 - \lambda_1(\Psi(0,k) + (t-t_k)\Omega(0,k))\right\}\Bigg(\xi_1\Bigg\{\sum_{i=0}^{m-1}\Big[{}_{t_i}I_{q_i}^{\alpha_i} f(s,x,{}_{t_i}I_{q_i}^{\beta_i}x)(t_{i+1})\\
&+\varphi_{i+1}\left(x(t_{i+1})\right)\Big] + \sum_{i=0}^{m-1}\Psi(i+1,m+1)\Big[{}_{t_i}I_{q_i}^{\alpha_i-\gamma_i} f(s,x,{}_{t_i}I_{q_i}^{\beta_i}x)(t_{i+1})\\
&+\varphi_{i+1}^*\left(x(t_{i+1})\right)\Big] + {}_{t_m}I_{q_m}^{\alpha_m} f(s,x,{}_{t_m}I_{q_m}^{\beta_m}x)(T)\Bigg\}\\
&+\xi_2\Bigg\{\sum_{i=0}^{m-1}\Omega(i+1,m+1)\left[{}_{t_i}I_{q_i}^{\alpha_i-\gamma_i} f(s,x,{}_{t_i}I_{q_i}^{\beta_i}x)(t_{i+1}) + \varphi_{i+1}^*\left(x(t_{i+1})\right)\right]\\
&+{}_{t_m}I_{q_m}^{\alpha_m-\gamma_m} f(s,x,{}_{t_m}I_{q_m}^{\beta_m}x)(T)\Bigg\}\Bigg) + \sum_{i=0}^{k-1}\left[{}_{t_i}I_{q_i}^{\alpha_i} f(s,x,{}_{t_i}I_{q_i}^{\beta_i}x)(t_{i+1}) + \varphi_{i+1}\left(x(t_{i+1})\right)\right]\\
&+\sum_{i=0}^{k-2}\Psi(i+1,k)\left[{}_{t_i}I_{q_i}^{\alpha_i-\gamma_i} f(s,x,{}_{t_i}I_{q_i}^{\beta_i}x)(t_{i+1}) + \varphi_{i+1}^*\left(x(t_{i+1})\right)\right]\\
&+(t-t_k)\sum_{i=0}^{k-1}\Omega(i+1,k)\left[{}_{t_i}I_{q_i}^{\alpha_i-\gamma_i} f(s,x,{}_{t_i}I_{q_i}^{\beta_i}x)(t_{i+1}) + \varphi_{i+1}^*\left(x(t_{i+1})\right)\right]\\
&+{}_{t_k}I_{q_k}^{\alpha_k} f(s,x,{}_{t_k}I_{q_k}^{\beta_k}x)(t),
\end{aligned}
\tag{4.1}
$$

where

$$ {}_aI_q^p f(s,x,{}_aI_q^\beta x)(u) = \frac{1}{\Gamma_q(p)} \int_a^u {}_a(u - {}_a\Phi_q(s))_q^{(p-1)} f(s,x(s),{}_aI_q^\beta x(s))_a d_q s, $$

with $p \in \{\alpha_0,\dots,\alpha_m,\alpha_0-\gamma_0,\dots,\alpha_m-\gamma_m\}$, $\beta \in \{\beta_0,\dots,\beta_m\}$, $q \in \{q_0,\dots,q_m\}$, $a \in \{t_0,\dots,t_m\}$ and $u \in \{t,t_1,t_2,\dots,t_m,T\}$.

In view of Lemma 3.2.1, we define an operator $\mathcal{A} : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ by

$$ \begin{aligned} \mathcal{A}x(t) \;=\; & -\frac{1}{2}\sum_{i=1}^{m}\left[{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}} f(t_i,x(t_i)) + \varphi_i\left({}_{t_{i-1}}I_{q_{i-1}}^{\beta_{i-1}} x(t_i)\right)\right] \\ & -\frac{1}{2}\sum_{i=1}^{m}(T-t_i)\left\{{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1} f(t_i,x(t_i)) + \varphi_i^*\left({}_{t_{i-1}}I_{q_{i-1}}^{\gamma_{i-1}} x(t_i)\right)\right\} \\ & -\frac{1}{2}{}_{t_m}I_{q_m}^{\alpha_m} f(T,x(T)) + \left(t - \frac{T}{2}\right)\left[-\frac{1}{2}\sum_{i=1}^{m}\left\{{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1} f(t_i,x(t_i))\right.\right. \\ & \left.\left. +\varphi_i^*\left({}_{t_{i-1}}I_{q_{i-1}}^{\gamma_{i-1}} x(t_i)\right)\right\} - \frac{1}{2}{}_{t_m}I_{q_m}^{\alpha_m-1} f(T,x(T))\right] \\ & +\sum_{i=1}^{k}\left[{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}} f(t_i,x(t_i)) + \varphi_i\left({}_{t_{i-1}}I_{q_{i-1}}^{\beta_{i-1}} x(t_i)\right)\right] \\ & +\sum_{i=1}^{k}(t-t_i)\left\{{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1} f(t_i,x(t_i)) + \varphi_i^*\left({}_{t_{i-1}}I_{q_{i-1}}^{\gamma_{i-1}} x(t_i)\right)\right\} \\ & +{}_{t_k}I_{q_k}^{\alpha_k} f(t,x(t)), \end{aligned} \tag{4.2} $$

where

$$ {}_aI_q^p f(u,x(u)) = \frac{1}{\Gamma_q(p)} \int_a^u {}_a(u - {}_a\Phi_q(s))_q^{(p-1)} f(s,x(s))_a d_q s, $$

$p \in \{\alpha_0,\dots,\alpha_m,\alpha_0-1,\dots,\alpha_m-1,\beta_0,\dots,\beta_{m-1},\gamma_0,\dots,\gamma_{m-1}\}$, $q \in \{q_0,\dots,q_m\}$, $a \in \{t_0,\dots,t_m\}$ and $u \in \{t,t_1,t_2,\dots,t_m,T\}$.

4.1 Existence results for BVP. (1.4)

Now we are in a position to present our main results. The first one is based on Krasnoselskii's fixed point theorem. Further, we set the notations:

$$\begin{aligned}
\Lambda_1 &= \frac{1}{|\Delta|}\{|\lambda_2|+|\lambda_1|\Psi(0,m+1)\} \\
&\quad\times\Bigg(|\xi_1|\Bigg\{\sum_{i=0}^{m}\frac{(t_{i+1}-t_i)^{\alpha_i}}{\Gamma_{q_i}(\alpha_i+1)}+\sum_{i=0}^{m-1}\Psi(i+1,m+1)\frac{(t_{i+1}-t_i)^{\alpha_i-\gamma_i}}{\Gamma_{q_i}(\alpha_i-\gamma_i+1)}\Bigg\} \\
&\quad+|\xi_2|\Bigg\{\sum_{i=0}^{m}\Omega(i+1,m+1)\frac{(t_{i+1}-t_i)^{\alpha_i-\gamma_i}}{\Gamma_{q_i}(\alpha_i-\gamma_i+1)}\Bigg\}\Bigg) \\
&\quad+\sum_{i=0}^{m}\frac{(t_{i+1}-t_i)^{\alpha_i}}{\Gamma_{q_i}(\alpha_i+1)}+\sum_{i=0}^{m-1}\Psi(i+1,m+1)\frac{(t_{i+1}-t_i)^{\alpha_i-\gamma_i}}{\Gamma_{q_i}(\alpha_i-\gamma_i+1)}, \qquad (4.3) \\
\Lambda_2(U) &= \frac{1}{|\Delta|}\{|\lambda_2|+|\lambda_1|\Psi(0,m+1)\} \\
&\quad\times\Bigg(|\xi_1|\Bigg\{mU_1+U_2\sum_{i=1}^{m}\Psi(i,m+1)\Bigg\}+|\xi_2|U_2\sum_{i=1}^{m}\Omega(i,m+1)\Bigg) \\
&\quad+mU_1+U_2\sum_{i=1}^{m}\Psi(i,m+1), \qquad (4.4)
\end{aligned}$$

where $U\in\{N,L\}$.

Theorem 4.1.1 *Let $f: J\times\mathbb{R}^2\to\mathbb{R}$ and $\varphi_k,\varphi_k^*:\mathbb{R}\to\mathbb{R}$, $k=1,\ldots,m$ be continuous functions. Assume that:*

(H_1) $|f(t,x,y)|\le\mu(t)$, $\forall(t,x,y)\in J\times\mathbb{R}^2$, *and* $\mu\in C(J,\mathbb{R}^+)$.

(H_2) *There exist positive constants L_1, L_2 such that $|\varphi_k(x)-\varphi_k(y)|\le L_1|x-y|$ and $|\varphi_k^*(x)-\varphi_k^*(y)|\le L_2|x-y|$ for each $x,y\in\mathbb{R}$, $k=1,2,\ldots,m$.*

(H_3) *There exist positive constants N_1,N_2 such that $|\varphi_k(x)|\le N_1$ and $|\varphi_k^*(x)|\le N_2$ for all $x\in\mathbb{R}$, for $k=1,2,\ldots,m$.*

Then problem (1.4) *has at least one solution on J provided that*

$$\Lambda_2(L)<1. \qquad (4.5)$$

Proof. Let us define $\sup_{t\in J}|\mu(t)|=\|\mu\|$ and select a suitable ball $B_R=\{x\in PC(J,\mathbb{R}):\|x\|_{PC}\le R\}$, where

$$R\ge\|\mu\|\Lambda_1+\Lambda_2(N), \qquad (4.6)$$

and Λ_1, Λ_2 are defined by (4.3) and (4.4) respectively. Next we define the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on B_R for $t \in J$ as

$$\begin{aligned}(\mathcal{Q}_1 x)(t) &= \frac{1}{\Delta}\{\lambda_2 - \lambda_1(\Psi(0,k) + (t-t_k)\Omega(0,k))\}\Bigg(\xi_1\Bigg\{\sum_{i=0}^{m} {}_{t_i}I_{q_i}^{\alpha_i} f(s,x,{}_{t_i}I_{q_i}^{\beta_i}x)(t_{i+1}) \\ &\quad + \sum_{i=0}^{m-1}\Psi(i+1,m+1)_{t_i}I_{q_i}^{\alpha_i-\gamma_i} f(s,x,{}_{t_i}I_{q_i}^{\beta_i}x)(t_{i+1})\Bigg\} \\ &\quad + \xi_2\Bigg\{\sum_{i=0}^{m}\Omega(i+1,m+1)_{t_i}I_{q_i}^{\alpha_i-\gamma_i} f(s,x,{}_{t_i}I_{q_i}^{\beta_i}x)(t_{i+1})\Bigg\}\Bigg) \\ &\quad + \sum_{i=0}^{k-1} {}_{t_i}I_{q_i}^{\alpha_i} f(s,x,{}_{t_i}I_{q_i}^{\beta_i}x)(t_{i+1}) + \sum_{i=0}^{k-2}\Psi(i+1,k)_{t_i}I_{q_i}^{\alpha_i-\gamma_i} f(s,x,{}_{t_i}I_{q_i}^{\beta_i}x)(t_{i+1}) \\ &\quad + (t-t_k)\sum_{i=0}^{k-1}\Omega(i+1,k)_{t_i}I_{q_i}^{\alpha_i-\gamma_i} f(s,x,{}_{t_i}I_{q_i}^{\beta_i}x)(t_{i+1}) + {}_{t_k}I_{q_k}^{\alpha_k} f(s,x,{}_{t_k}I_{q_k}^{\beta_k}x)(t),\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}(\mathcal{Q}_2 x)(t) &= \frac{1}{\Delta}\{\lambda_2 - \lambda_1(\Psi(0,k) + (t-t_k)\Omega(0,k))\}\Bigg(\xi_1\Bigg\{\sum_{i=1}^{m}\varphi_i(x(t_i)) \\ &\quad + \sum_{i=1}^{m}\Psi(i,m+1)\varphi_i^*(x(t_i))\Bigg\} + \xi_2\Bigg\{\sum_{i=1}^{m}\Omega(i,m+1)\varphi_i^*(x(t_i))\Bigg\}\Bigg) \\ &\quad + \sum_{i=1}^{k}\varphi_i(x(t_i)) + \sum_{i=1}^{k-1}\Psi(i,k)\varphi_i^*(x(t_i)) + (t-t_k)\sum_{i=1}^{k}\Omega(i,k)\varphi_i^*(x(t_i)).\end{aligned} \tag{4.8}$$

For any $x, y \in B_R$, we have

$$
\begin{aligned}
& |\mathcal{Q}_1 x(t) + \mathcal{Q}_2 y(t)| \\
\leq\ & \frac{\|\mu\|}{|\Delta|}\{|\lambda_2| + |\lambda_1|(\Psi(0,m) + (T - t_m)\Omega(0,m))\} \\
& \times\Bigg(|\xi_1|\Bigg\{\sum_{i=0}^{m} {}_{t_i}I_{q_i}^{\alpha_i}1(t_{i+1}) + \sum_{i=0}^{m-1}\Psi(i+1,m+1)_{t_i}I_{q_i}^{\alpha_i-\gamma_i}1(t_{i+1})\Bigg\} \\
& + |\xi_2|\Bigg\{\sum_{i=0}^{m}\Omega(i+1,m+1)_{t_i}I_{q_i}^{\alpha_i-\gamma_i}1(t_{i+1})\Bigg\}\Bigg) \\
& + \|\mu\|\sum_{i=0}^{m-1} {}_{t_i}I_{q_i}^{\alpha_i}1(t_{i+1}) + \|\mu\|\sum_{i=0}^{m-2}\Psi(i+1,m)_{t_i}I_{q_i}^{\alpha_i-\gamma_i}1(t_{i+1}) \\
& + \|\mu\|(T - t_m)\sum_{i=0}^{m-1}\Omega(i+1,m)_{t_i}I_{q_i}^{\alpha_i-\gamma_i}1(t_{i+1}) + \|\mu\|_{t_m}I_{q_m}^{\alpha_m}1(T) \\
& + \frac{1}{|\Delta|}\{|\lambda_2| + |\lambda_1|(\Psi(0,m) + (T - t_m)\Omega(0,m))\} \\
& \times\left(|\xi_1|\left\{mN_1 + N_2\sum_{i=1}^{m}\Psi(i,m+1)\right\} + |\xi_2|N_2\sum_{i=1}^{m}\Omega(i,m+1)\right) \\
& + mN_1 + N_2\sum_{i=1}^{m-1}\Psi(i,m) + N_2(T - t_m)\sum_{i=1}^{m}\Omega(i,m) \\
=\ & \frac{\|\mu\|}{|\Delta|}\{|\lambda_2| + |\lambda_1|\Psi(0,m+1)\}\Bigg(|\xi_1|\Bigg\{\sum_{i=0}^{m}\frac{(t_{i+1} - t_i)^{\alpha_i}}{\Gamma_{q_i}(\alpha_i + 1)} \\
& + \sum_{i=0}^{m-1}\Psi(i+1,m+1)\frac{(t_{i+1} - t_i)^{\alpha_i-\gamma_i}}{\Gamma_{q_i}(\alpha_i - \gamma_i + 1)}\Bigg\} \\
& + |\xi_2|\Bigg\{\sum_{i=0}^{m}\Omega(i+1,m+1)\frac{(t_{i+1} - t_i)^{\alpha_i-\gamma_i}}{\Gamma_{q_i}(\alpha_i - \gamma_i + 1)}\Bigg\}\Bigg) \\
& + \|\mu\|\sum_{i=0}^{m-1}\frac{(t_{i+1} - t_i)^{\alpha_i}}{\Gamma_{q_i}(\alpha_i + 1)} + \|\mu\|\sum_{i=0}^{m-2}\Psi(i+1,m)\frac{(t_{i+1} - t_i)^{\alpha_i-\gamma_i}}{\Gamma_{q_i}(\alpha_i - \gamma_i + 1)} \\
& + \|\mu\|(T - t_m)\sum_{i=0}^{m-1}\Omega(i+1,m)\frac{(t_{i+1} - t_i)^{\alpha_i-\gamma_i}}{\Gamma_{q_i}(\alpha_i - \gamma_i + 1)} + \|\mu\|\frac{(T - t_m)^{\alpha_m}}{\Gamma_{q_m}(\alpha_m + 1)} + \Lambda_2(N) \\
=\ & \|\mu\|\Lambda_1 + \Lambda_2(N),
\end{aligned}
$$

which implies $\|\mathcal{Q}_1 x + \mathcal{Q}_2 y\| \leq \|\mu\|\Lambda_1 + \Lambda_2(N)$. Therefore, $\mathcal{Q}_1 x + \mathcal{Q}_2 y \in B_R$. This shows that condition (a) of Theorem 4.1.1 is satisfied.

Next we will show that the operator $\mathcal{Q}_1$ satisfies condition (b) of Theorem 4.1.1. Using the earlier arguments, we get that $\mathcal{Q}_1$ is uniformly bounded on B_R, that is,

$$\|\mathcal{Q}_1 x\|_{PC} \leq \|\mu\|\Lambda_1.$$

Now we establish the compactness of $\mathcal{Q}_1$. Setting $\sup_{(t,x,y)\in J\times B_R\times B_R}|f(t,x,y)| = \mu^* < \infty$, then, for each $\tau_1, \tau_2 \in (t_k, t_{k+1})$ for some $k \in \{0, 1, \ldots, m\}$ with $\tau_2 > \tau_1$, we obtain

$$\begin{aligned}
|(\mathcal{Q}_1 x)(\tau_2) - (\mathcal{Q}_1 x)(\tau_1)| \leq\ & |\tau_2 - \tau_1| \frac{|\lambda_1|\mu^*}{|\Delta|}\Omega(0,k)\left(|\xi_1|\left\{\sum_{i=0}^{m}\frac{(t_{i+1}-t_i)^{\alpha_i}}{\Gamma_{q_i}(\alpha_i+1)}\right.\right. \\
&\left. + \sum_{i=0}^{m-1}\Psi(i+1,m+1)\frac{(t_{i+1}-t_i)^{\alpha_i-\gamma_i}}{\Gamma_{q_i}(\alpha_i-\gamma_i+1)}\right\} \\
&\left. + |\xi_2|\left\{\sum_{i=0}^{m}\Omega(i+1,m+1)\frac{(t_{i+1}-t_i)^{\alpha_i-\gamma_i}}{\Gamma_{q_i}(\alpha_i-\gamma_i+1)}\right\}\right) \\
&+ \mu^*|\tau_2-\tau_1|\sum_{i=0}^{k-1}\Omega(i+1,k)\frac{(t_{i+1}-t_i)^{\alpha_i-\gamma_i}}{\Gamma_{q_i}(\alpha_i-\gamma_i+1)} \\
&+ \frac{\mu^*}{\Gamma_{q_k}(\alpha_k+1)}\left|(\tau_2-t_k)^{\alpha_k} - (\tau_1-t_k)^{\alpha_k}\right|.
\end{aligned}$$

As $\tau_1 \to \tau_2$, the right hand side of the above inequality tends to zero (independent of x). Therefore, the operator $\mathcal{Q}_1$ is equicontinuous. Since $\mathcal{Q}_1$ maps bounded subsets into relatively compact subsets, it follows that $\mathcal{Q}_1$ is relative compact on B_R. Hence, by the Arzelá-Ascoli theorem, $\mathcal{Q}_1$ is compact on B_R. Now let $x_n \in B_R$ with $\|x_n - x\| \to 0$, $n \to \infty$. Then the limit $|x_n(t) - x(t)| \to 0$ uniformly valid on J. From the uniform continuity of $f(t, x, {}_{t_k}I_{q_k}^{\beta_k}x)$ on the compact set $J \times [-R, R] \times [-R, R]$ if follows that $|f(t, x_n, {}_{t_k}I_{q_k}^{\beta_k}x_n) - f(t, x, {}_{t_k}I_{q_k}^{\beta_k}x)| \to 0$, $n \to \infty$, is uniformly valid on J. Hence $\|\mathcal{Q}_1 x_n - \mathcal{Q}_1 x\|_{PC} \to 0$ as $n \to \infty$ which proves the continuity of $\mathcal{Q}_1$.

Now we show that $\mathcal{Q}_2$ is a contraction. For $x, y \in PC(J, \mathbb{R})$, we have

$$\begin{aligned}
& |\mathcal{Q}_2 x(t) - \mathcal{Q}_2 y(t)| \\
\leq\ & \frac{1}{|\Delta|}\{|\lambda_2| + |\lambda_1|\Psi(0, m+1)\} \\
& \times\Bigg(|\xi_1|\Bigg\{\sum_{i=1}^{m}|\varphi_i(x(t_i)) - \varphi_i(y(t_i))| + \sum_{i=1}^{m}\Psi(i, m+1)|\varphi_i^*(x(t_i)) - \varphi_i^*(y(t_i))|\Bigg\} \\
& +|\xi_2|\sum_{i=1}^{m}\Omega(i, m+1)|\varphi_i^*(x(t_i)) - \varphi_i^*(y(t_i))|\Bigg) + \sum_{i=1}^{m}|\varphi_i(x(t_i)) - \varphi_i(y(t_i))| \\
& + \sum_{i=1}^{m}\Psi(i, m+1)|\varphi_i^*(x(t_i)) - \varphi_i^*(y(t_i))| \\
\leq\ & \Lambda_2(L)\|x - y\|_{PC},
\end{aligned}$$

which yields $\|\mathcal{Q}_2 x - \mathcal{Q}_2 y\|_{PC} \leq \Lambda_2(L)\|x - y\|_{PC}$. From (4.5), it follows that $\mathcal{Q}_\in$ is a contraction. Thus problem (1.4) has at least one solution on J. The proof is completed. □

Our next result is based on a fixed point theorem due to O'Regan. For the sake of brevity, we use the following constants in the sequel.

$$\begin{aligned}
\Lambda_3 =\ & \frac{1}{|\Delta|}\{|\lambda_2| + |\lambda_1|\Psi(0, m+1)\} \\
& \times\Bigg(|\xi_1|\Bigg\{\sum_{i=0}^{m}\frac{(t_{i+1} - t_i)^{\alpha_i+\beta_i}}{\Gamma_{q_i}(\alpha_i + \beta_i + 1)} + \sum_{i=0}^{m-1}\Psi(i+1, m+1)\frac{(t_{i+1} - t_i)^{\alpha_i+\beta_i-\gamma_i}}{\Gamma_{q_i}(\alpha_i + \beta_i - \gamma_i + 1)}\Bigg\} \\
& + |\xi_2|\Bigg\{\sum_{i=0}^{m}\Omega(i+1, m+1)\frac{(t_{i+1} - t_i)^{\alpha_i+\beta_i-\gamma_i}}{\Gamma_{q_i}(\alpha_i + \beta_i - \gamma_i + 1)}\Bigg\}\Bigg) \\
& + \sum_{i=0}^{m}\frac{(t_{i+1} - t_i)^{\alpha_i+\beta_i}}{\Gamma_{q_i}(\alpha_i + \beta_i + 1)} + \sum_{i=0}^{m-1}\Psi(i+1, m+1)\frac{(t_{i+1} - t_i)^{\alpha_i+\beta_i-\gamma_i}}{\Gamma_{q_i}(\alpha_i + \beta_i - \gamma_i + 1)}, \qquad (4.9)\\
\Lambda_4 =\ & \frac{1}{|\Delta|}\{|\lambda_2| + |\lambda_1|\Psi(0, m+1)\}m|\xi_1| + m, \qquad (4.10)\\
\Lambda_5 =\ & \frac{1}{|\Delta|}\{|\lambda_2| + |\lambda_1|\Psi(0, m+1)\}\sum_{i=1}^{m}(|\xi_1|\Psi(i, m+1) + |\xi_2|\Omega(i, m+1)) \\
& + \sum_{i=1}^{m}\Psi(i, m+1). \qquad (4.11)
\end{aligned}$$

Theorem 4.1.2 *Suppose that the condition* (H_3) *and the following assumptions hold:*

(H_4) *There exist a continuous nondecreasing function* $\psi : [0, \infty) \to (0, \infty)$ *and a continuous function* $p : J \to \mathbb{R}^+$ *such that*

$$|f(t, x, y)| \leq p(t)\psi(|x|) + |y|, \quad \forall (t, x, y) \in J \times \mathbb{R}^{\nvDash}. \tag{4.12}$$

(H_5) *There exist two continuous nondecreasing functions* $\omega_1, \omega_2 : [0, \infty) \to [0, \infty)$ *and two positive constants* D_1, D_2 *such that*

$$\begin{aligned} |\varphi_k(x) - \varphi_k(y)| &\leq \omega_1(|x - y|) \quad \text{and} \quad \omega_1(|x|) \leq D_1|x|, & (4.13)\\ |\varphi_k^*(x) - \varphi_k^*(y)| &\leq \omega_2(|x - y|) \quad \text{and} \quad \omega_2(|x|) \leq D_2|x|, & (4.14) \end{aligned}$$

for all $x, y \in \mathbb{R}$, *for* $k = 1, 2, \ldots, m$ *satisfying*

$$D_1\Lambda_4 + D_2\Lambda_5 < 1. \tag{4.15}$$

where Λ_4, Λ_5 *are defined by* (4.10) *and* (4.11), *respectively.*

(H_6)

$$\sup_{r \in (0,\infty)} \frac{r}{p^*\psi(r)\Lambda_1 + \Lambda_2(N)} > \frac{1}{1 - \Lambda_3}, \quad \Lambda_3 < 1, \tag{4.16}$$

where $p^* = \sup_{t \in J} |p(t)|$, Λ_1, $\Lambda_2(N)$ *and* Λ_3 *are defined by* (4.3)-(4.4) *and* (4.9), *respectively.*

Then there exists at least one solution for problem (1.4) *on* J.

Proof. Consider the operator $\mathcal{Q} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ defined by (4.1) as

$$\mathcal{Q}x(t) = \mathcal{Q}_1x(t) + \mathcal{Q}_2x(t), \quad t \in J,$$

where $\mathcal{Q}_1$, $\mathcal{Q}_2$ are given by (4.7) and (4.8) respectively. From (H_6), there exists a positive constant $\rho > 0$ such that

$$\frac{\rho}{p^*\psi(\rho)\Lambda_1 + \Lambda_2(N)} > \frac{1}{1 - \Lambda_3}.$$

Let $B_\rho = \{x \in PC : \|x\|_{PC} \leq \rho\}$. As in the proof of Theorem 4.1.1, $\mathcal{Q}_1$ is continuous. Using (4.12), we now show that $\mathcal{Q}_1(B_\rho)$ is bounded. For any

$x \in B_\rho$, we have

$$
\begin{aligned}
|\mathcal{Q}_1 x(t)| \leq\ & \frac{1}{|\Delta|}\left\{|\lambda_2| + |\lambda_1|\Psi(0,m+1)\right\}\Bigg(|\xi_1|\Bigg\{\sum_{i=0}^{m} {}_{t_i}I_{q_i}^{\alpha_i}\left(p^*\psi(\rho) + {}_{t_i}I_{q_i}^{\beta_i}\rho\right)(t_{i+1}) \\
& + \sum_{i=0}^{m-1}\Psi(i+1,m+1){}_{t_i}I_{q_i}^{\alpha_i-\gamma_i}\left(p^*\psi(\rho) + {}_{t_i}I_{q_i}^{\beta_i}\rho\right)(t_{i+1})\Bigg\} \\
& + |\xi_2|\Bigg\{\sum_{i=0}^{m}\Omega(i+1,m+1){}_{t_i}I_{q_i}^{\alpha_i-\gamma_i}\left(p^*\psi(\rho) + {}_{t_i}I_{q_i}^{\beta_i}\rho\right)(t_{i+1})\Bigg\}\Bigg) \\
& + \sum_{i=0}^{m} {}_{t_i}I_{q_i}^{\alpha_i}\left(p^*\psi(\rho) + {}_{t_i}I_{q_i}^{\beta_i}\rho\right)(t_{i+1}) \\
& + \sum_{i=0}^{m-1}\Psi(i+1,m+1){}_{t_i}I_{q_i}^{\alpha_i-\gamma_i}\left(p^*\psi(\rho) + {}_{t_i}I_{q_i}^{\beta_i}\rho\right)(t_{i+1}) \\
=\ & \frac{1}{|\Delta|}\left\{|\lambda_2| + |\lambda_1|\Psi(0,m+1)\right\} \\
& \times\Bigg(|\xi_1|\Bigg\{p^*\psi(\rho)\sum_{i=0}^{m} {}_{t_i}I_{q_i}^{\alpha_i}1(t_{i+1}) + \rho\sum_{i=0}^{m} {}_{t_i}I_{q_i}^{\alpha_i+\beta_i}1(t_{i+1}) \\
& + p^*\psi(\rho)\sum_{i=0}^{m-1}\Psi(i+1,m+1){}_{t_i}I_{q_i}^{\alpha_i-\gamma_i}1(t_{i+1}) \\
& + \rho\sum_{i=0}^{m-1}\Psi(i+1,m+1){}_{t_i}I_{q_i}^{\alpha_i+\beta_i-\gamma_i}1(t_{i+1})(t_{i+1})\Bigg\} \\
& + |\xi_2|\Bigg\{p^*\psi(\rho)\sum_{i=0}^{m}\Omega(i+1,m+1){}_{t_i}I_{q_i}^{\alpha_i-\gamma_i}1(t_{i+1}) \\
& + \rho\sum_{i=0}^{m}\Omega(i+1,m+1){}_{t_i}I_{q_i}^{\alpha_i+\beta_i-\gamma_i}1(t_{i+1})\Bigg\}\Bigg) \\
& + p^*\psi(\rho)\sum_{i=0}^{m} {}_{t_i}I_{q_i}^{\alpha_i}1(t_{i+1}) + \rho\sum_{i=0}^{m} {}_{t_i}I_{q_i}^{\alpha_i+\beta_i}1(t_{i+1}) \\
& + p^*\psi(\rho)\sum_{i=0}^{m-1}\Psi(i+1,m+1){}_{t_i}I_{q_i}^{\alpha_i-\gamma_i}1(t_{i+1}) \\
& + \rho\sum_{i=0}^{m-1}\Psi(i+1,m+1){}_{t_i}I_{q_i}^{\alpha_i+\beta_i-\gamma_i}1(t_{i+1}) \\
=\ & p^*\psi(\rho)\Lambda_1 + \rho\Lambda_3.
\end{aligned}
$$

Thus $\mathcal{Q}_1$ is uniformly bounded. As in the proof of Theorem 4.1.1, we can show that $\mathcal{Q}_1$ is equicontinuous. In consequence, it follows by Arzelá-Ascoli theorem that $\mathcal{Q}_1(B_\rho)$ is compact and hence completely continuous.

Next, we will show that $\mathcal{Q}_2$ is a nonlinear contraction. Define a continuous nondecreasing function $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\nu(\varepsilon) = (D_1\Lambda_4 + D_2\Lambda_5)\varepsilon, \quad \forall \varepsilon \geq 0.$$

Note that $\nu(0) = 0$ and, by condition (4.15), $\nu(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$. For any $x, y \in B_\rho$, we have

$$\begin{aligned}
& |\mathcal{Q}_2x(t) - \mathcal{Q}_2y(t)| \\
\leq\ & \frac{1}{\Delta}\{|\lambda_2| + |\lambda_1|\Psi(0,m+1)\}\Bigg(|\xi_1|\Bigg\{\sum_{i=1}^{m}|\varphi_i(x(t_i)) - \varphi_i(y(t_i))| \\
& + \sum_{i=1}^{m}\Psi(i,m+1)|\varphi_i^*(x(t_i)) - \varphi_i^*(y(t_i))|\Bigg\} \\
& + |\xi_2|\left\{\sum_{i=1}^{m}\Omega(i,m+1)|\varphi_i^*(x(t_i)) - \varphi_i^*(y(t_i))|\right\}\Bigg) \\
& + \sum_{i=1}^{m}|\varphi_i(x(t_i)) - \varphi_i(y(t_i))| + \sum_{i=1}^{m}\Psi(i,m+1)|\varphi_i^*(x(t_i)) - \varphi_i^*(y(t_i))| \\
\leq\ & \frac{1}{\Delta}\{|\lambda_2| + |\lambda_1|\Psi(0,m+1)\}\Bigg(|\xi_1|\Bigg\{\sum_{i=1}^{m}\omega_1(\|x-y\|_{PC}) \\
& + \sum_{i=1}^{m}\Psi(i,m+1)\omega_2(\|x-y\|_{PC})\Bigg\} + |\xi_2|\left\{\sum_{i=1}^{m}\Omega(i,m+1)\omega_2(\|x-y\|_{PC})\right\}\Bigg) \\
& + \sum_{i=1}^{m}\omega_1(\|x-y\|_{PC}) + \sum_{i=1}^{m}\Psi(i,m+1)\omega_2(\|x-y\|_{PC}) \\
\leq\ & \nu(\|x-y\|_{PC}).
\end{aligned}$$

Thus $\|\mathcal{Q}_2x - \mathcal{Q}_2y\|_{PC} \leq \nu(\|x-y\|_{PC})$ which implies that $\mathcal{Q}_2$ is nonlinear contraction.

Next, we will show that the set $\mathcal{Q}(B_\rho)$ is bounded. Using (H_3), for $x \in B_\rho$,, we have

$$
\begin{aligned}
& |\mathcal{Q}_2 x(t)| \\
\leq\ & \frac{1}{\Delta}\left\{|\lambda_2| + |\lambda_1|\Psi(0,m+1)\right\}\left(|\xi_1|\left\{\sum_{i=1}^{m}|\varphi_i\left(x(t_i)\right)| + \sum_{i=1}^{m}\Psi(i,m+1)|\varphi_i^*\left(x(t_i)\right)|\right\}\right. \\
& \left. +|\xi_2|\left\{\sum_{i=1}^{m}\Omega(i,m+1)|\varphi_i^*\left(x(t_i)\right)|\right\}\right) + \sum_{i=1}^{m}|\varphi_i\left(x(t_i)\right)| + \sum_{i=1}^{m}\Psi(i,m+1)|\varphi_i^*\left(x(t_i)\right)| \\
\leq\ & \frac{1}{\Delta}\left\{|\lambda_2| + |\lambda_1|\Psi(0,m+1)\right\}\left(|\xi_1|\left\{\sum_{i=1}^{m}N_1 + \sum_{i=1}^{m}\Psi(i,m+1)N_2\right\}\right. \\
& \left. +|\xi_2|\left\{\sum_{i=1}^{m}\Omega(i,m+1)N_2\right\}\right) + \sum_{i=1}^{m}N_1 + \sum_{i=1}^{m}\Psi(i,m+1)N_2 \\
=\ & \Lambda_2(N),
\end{aligned}
$$

which together with the boundedness of the set $\mathcal{Q}_1(B_\rho)$ implies that the set $\mathcal{Q}(B_\rho)$ is bounded.

Finally, it will be shown that the case $(C2)$ of Theorem Oregan is false. On the contrary, we suppose that $(C2)$ holds true. Then, there exists $\theta \in (0,1)$ and

$x \in \partial B_\rho$ such that $x = \theta \mathcal{Q}x$. Thus, we have $\|x\|_{PC} = \rho$ and

$$\begin{aligned}
|x(t)| &= \theta|\mathcal{Q}_1 x(t) + \mathcal{Q}_2 x(t)| \\
&\leq |\mathcal{Q}_1 x(t)| + |\mathcal{Q}_2 x(t)| \\
&\leq \frac{1}{|\Delta|}\{|\lambda_2| + |\lambda_1|\Psi(0, m+1)\}\Bigg(|\xi_1|\Bigg\{\sum_{i=0}^{m} {}_{t_i}I_{q_i}^{\alpha_i}\left(p^*\psi(\rho) + {}_{t_i}I_{q_i}^{\beta_i}\rho\right)(t_{i+1}) \\
&\quad + \sum_{i=0}^{m-1}\Psi(i+1, m+1){}_{t_i}I_{q_i}^{\alpha_i-\gamma_i}\left(p^*\psi(\rho) + {}_{t_i}I_{q_i}^{\beta_i}\rho\right)(t_{i+1})\Bigg\} \\
&\quad + |\xi_2|\Bigg\{\sum_{i=0}^{m}\Omega(i+1, m+1){}_{t_i}I_{q_i}^{\alpha_i-\gamma_i}\left(p^*\psi(\rho) + {}_{t_i}I_{q_i}^{\beta_i}\rho\right)(t_{i+1})\Bigg\}\Bigg) \\
&\quad + \sum_{i=0}^{m} {}_{t_i}I_{q_i}^{\alpha_i}\left(p^*\psi(\rho) + {}_{t_i}I_{q_i}^{\beta_i}\rho\right)(t_{i+1}) \\
&\quad + \sum_{i=0}^{m-1}\Psi(i+1, m+1){}_{t_i}I_{q_i}^{\alpha_i-\gamma_i}\left(p^*\psi(\rho) + {}_{t_i}I_{q_i}^{\beta_i}\rho\right)(t_{i+1}) \\
&\quad + \frac{1}{\Delta}\{|\lambda_2| + |\lambda_1|\Psi(0, m+1)\}\Bigg(|\xi_1|\Bigg\{\sum_{i=1}^{m} N_1 + \sum_{i=1}^{m}\Psi(i, m+1)N_2\Bigg\} \\
&\quad + |\xi_2|\Bigg\{\sum_{i=1}^{m}\Omega(i, m+1)N_2\Bigg\}\Bigg) + \sum_{i=1}^{m} N_1 + \sum_{i=1}^{m}\Psi(i, m+1)N_2 \\
&\leq p^*\psi(\rho)\Lambda_1 + \rho\Lambda_3 + \Lambda_2(N),
\end{aligned}$$

which implies that

$$\rho \leq p^*\psi(\rho)\Lambda_1 + \rho\Lambda_3 + \Lambda_2(N).$$

This can alternatively be written as

$$\frac{\rho}{p^*\psi(\rho)\Lambda_1 + \Lambda_2(N)} \leq \frac{1}{1 - \Lambda_3},$$

which contradicts (H_6). Thus the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ satisfy all the conditions of Theorem Oregan. Therefore, by the conclusion of Theorem Oregan, problem (1.4) has at least one solution on J. This completes the proof. □

Finally we show the uniqueness of solutions of problem (1.4) by applying Banach's contraction mapping principle.

Theorem 4.1.3 *Let the condition* (H_2) *holds. Further, there exist functions* $M_1(t), M_2(t) \in C(J, \mathbb{R}^+)$ *such that*

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| \leq M_1(t)|x - \bar{x}| + M_2(t)|y - \bar{y}|, \ \forall t \in J, \ x, \bar{x}, y, \bar{y} \in \mathbb{R}. \quad (4.17)$$

Then problem (1.4) has a unique solution on J if

$$M_1^*\Lambda_1 + M_2^*\Lambda_3 + \Lambda_4 L_1 + \Lambda_5 L_2 < 1, \tag{4.18}$$

where $M_1^* = \sup_{t\in J}|M_1(t)|$ *and* $M_2^* = \sup_{t\in J}|M_2(t)$.

Proof. For any $x, y \in PC(J,\mathbb{R})$, we have

$$\begin{aligned}
& |\mathcal{Q}x(t) - \mathcal{Q}y(t)| \\
\leq\ & \frac{1}{|\Delta|}\{|\lambda_2| + |\lambda_1|\Psi(0,m+1)\} \\
& \times\Bigg(|\xi_1|\Bigg\{\sum_{i=0}^{m-1}\Big[{}_{t_i}I_{q_i}^{\alpha_i}\left|f(s,x,{}_{t_i}I_{q_i}^{\beta_i}x) - f(s,y,{}_{t_i}I_{q_i}^{\beta_i}y)\right|(t_{i+1}) \\
& + |\varphi_{i+1}(x(t_{i+1})) - \varphi_{i+1}(y(t_{i+1}))|\Big] \\
& + \sum_{i=0}^{m-1}\Psi(i+1,m+1)\Big[{}_{t_i}I_{q_i}^{\alpha_i-\gamma_i}\left|f(s,x,{}_{t_i}I_{q_i}^{\beta_i}x) - f(s,y,{}_{t_i}I_{q_i}^{\beta_i}y)\right|(t_{i+1}) \\
& + \left|\varphi_{i+1}^*(x(t_{i+1})) - \varphi_{i+1}^*(y(t_{i+1}))\right|\Big] + {}_{t_m}I_{q_m}^{\alpha_m}\left|f(s,x,{}_{t_m}I_{q_m}^{\beta_m}x) - f(s,y,{}_{t_m}I_{q_m}^{\beta_m}y)\right|(T)\Bigg\} \\
& + |\xi_2|\Bigg\{\sum_{i=0}^{m-1}\Omega(i+1,m+1)\Big[{}_{t_i}I_{q_i}^{\alpha_i-\gamma_i}\left|f(s,x,{}_{t_i}I_{q_i}^{\beta_i}x) - f(s,y,{}_{t_i}I_{q_i}^{\beta_i}y)\right|(t_{i+1}) \\
& + \left|\varphi_{i+1}^*(x(t_{i+1})) - \varphi_{i+1}^*(y(t_{i+1}))\right|\Big] \\
& + {}_{t_m}I_{q_m}^{\alpha_m-\gamma_m}\left|f(s,x,{}_{t_m}I_{q_m}^{\beta_m}x) - f(s,y,{}_{t_m}I_{q_m}^{\beta_m}y)\right|(T)\Bigg\}\Bigg) \\
& + \sum_{i=0}^{m-1}\Big[{}_{t_i}I_{q_i}^{\alpha_i}\left|f(s,x,{}_{t_i}I_{q_i}^{\beta_i}x) - f(s,y,{}_{t_i}I_{q_i}^{\beta_i}y)\right|(t_{i+1}) \\
& + |\varphi_{i+1}(x(t_{i+1})) - \varphi_{i+1}(y(t_{i+1}))|\Big] \\
& + \sum_{i=0}^{m-2}\Psi(i+1,m)\Big[{}_{t_i}I_{q_i}^{\alpha_i-\gamma_i}\left|f(s,x,{}_{t_i}I_{q_i}^{\beta_i}x) - f(s,y,{}_{t_i}I_{q_i}^{\beta_i}y)\right|(t_{i+1}) \\
& + \left|\varphi_{i+1}^*(x(t_{i+1})) - \varphi_{i+1}^*(y(t_{i+1}))\right|\Big] \\
& + (T-t_m)\sum_{i=0}^{m-1}\Omega(i+1,m)\Big[{}_{t_i}I_{q_i}^{\alpha_i-\gamma_i}\left|f(s,x,{}_{t_i}I_{q_i}^{\beta_i}x) - f(s,y,{}_{t_i}I_{q_i}^{\beta_i}y)\right|(t_{i+1}) \\
& + \left|\varphi_{i+1}^*(x(t_{i+1})) - \varphi_{i+1}^*(y(t_{i+1}))\right|\Big] \\
& + {}_{t_m}I_{q_m}^{\alpha_m}\left|f(s,x,{}_{t_m}I_{q_m}^{\beta_m}x) - f(s,x,{}_{t_m}I_{q_m}^{\beta_m}x)\right|(T) \\
\leq\ & (M_1^*\Lambda_1 + M_2^*\Lambda_3 + \Lambda_4 L_1 + \Lambda_5 L_2)\|x-y\|_{PC},
\end{aligned}$$

which implies that

$$\|\mathcal{Q}x - \mathcal{Q}y\|_{PC} \leq (M_1^*\Lambda_1 + M_2^*\Lambda_3 + \Lambda_4 L_1 + \Lambda_5 L_2)\|x - y\|_{PC}.$$

Thus the operator $\mathcal{Q}$ is a contraction in view of the condition (4.31). Consequently, by Banach's contraction mapping principle, problem (1.4) has a unique solution on J. The proof is completed. □

4.2 Existence results for BVP. (1.5)

For computational convenience, we set

$$\begin{aligned}\Omega_1 &= \frac{3}{2}\sum_{i=1}^{m+1}\frac{(t_i - t_{i-1})^{\alpha_{i-1}}}{\Gamma_{q_{i-1}}(\alpha_{i-1}+1)} + \frac{3}{2}\sum_{i=1}^{m}\frac{(T-t_i)(t_i - t_{i-1})^{\alpha_{i-1}-1}}{\Gamma_{q_{i-1}}(\alpha_{i-1})} \\ &\quad + \frac{T}{4}\sum_{i=1}^{m+1}\frac{(t_i - t_{i-1})^{\alpha_{i-1}-1}}{\Gamma_{q_{i-1}}(\alpha_{i-1})}, \end{aligned} \tag{4.19}$$

$$\Omega_2 = \frac{3}{2}mM_1 + \frac{3}{2}M_2\sum_{i=1}^{m}(T-t_i) + \frac{T}{4}mM_2. \tag{4.20}$$

Now we present our first existence result for the problem (1.5) which is based on Schauder fixed point theorem.

Theorem 4.2.1 *Assume that*

(H_1) *there exist continuous functions $a(t)$, $b(t)$ and nonnegative constants M_1, M_2 such that*

$$|f(t,x)| \leq a(t) + b(t)|x|, \quad (t,x) \in J \times \mathbb{R}, \tag{4.21}$$

with $\sup_{t\in J}|a(t)| = a_1$, $\sup_{t\in J}|b(t)| = b_1$ and

$$|\varphi_k(x)| \leq M_1, \quad |\varphi_k^*(x)| \leq M_2, \quad \forall x \in \mathbb{R},\ k = 1,2,\ldots,m. \tag{4.22}$$

Then the anti-periodic boundary value problem (1.5) *has at least one solution on J if*

$$b_1\Omega_1 < 1. \tag{4.23}$$

Proof. Let us define a closed ball $B_R = \{x \in PC(J,\mathbb{R}) : \|x\|_{PC} \leq R\}$ with

$$R > \frac{a_1\Omega_1 + \Omega_2}{1 - b_1\Omega_1},$$

where a_1, b_1 are defined in (H_1) and Ω_1, Ω_2 are respectively given by (4.19) and (4.20). Clearly B_R is a bounded, closed and convex subset of $PC(J,\mathbb{R})$. Now we show that the operator $\mathcal{A}:\mathcal{PC}(\mathcal{J},\mathbb{R})\to\mathcal{PC}(\mathcal{J},\mathbb{R})$ defined by (4.2) has a fixed point in the following two steps.

Step 1. $\mathcal{A}:B_R\to B_R$.

For any $x \in B_R$, using (2.7), we have

$$\begin{aligned}
|\mathcal{A}x(t)| \leq & \frac{1}{2}\sum_{i=1}^{m}\left[{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}}|f(t_i,x(t_i))| + \left|\varphi_i\left({}_{t_{i-1}}I_{q_{i-1}}^{\beta_{i-1}}x(t_i)\right)\right|\right] \\
& + \frac{1}{2}\sum_{i=1}^{m}(T-t_i)\left\{{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1}|f(t_i,x(t_i))| + \left|\varphi_i^*\left({}_{t_{i-1}}I_{q_{i-1}}^{\gamma_{i-1}}x(t_i)\right)\right|\right\} \\
& + \frac{1}{2}{}_{t_m}I_{q_m}^{\alpha_m}|f(T,x(T))| + \frac{T}{2}\left[\frac{1}{2}\sum_{i=1}^{m}\left\{{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1}|f(t_i,x(t_i))|\right.\right. \\
& \left.\left. + \left|\varphi_i^*\left({}_{t_{i-1}}I_{q_{i-1}}^{\gamma_{i-1}}x(t_i)\right)\right|\right\} + \frac{1}{2}{}_{t_m}I_{q_m}^{\alpha_m-1}|f(T,x(T))|\right] \\
& + \sum_{i=1}^{k}\left[{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}}|f(t_i,x(t_i))| + \left|\varphi_i\left({}_{t_{i-1}}I_{q_{i-1}}^{\beta_{i-1}}x(t_i)\right)\right|\right] \\
& + \sum_{i=1}^{k}(t-t_i)\left\{{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1}|f(t_i,x(t_i))| + \left|\varphi_i^*\left({}_{t_{i-1}}I_{q_{i-1}}^{\gamma_{i-1}}x(t_i)\right)\right|\right\} \\
& + {}_{t_k}I_{q_k}^{\alpha_k}|f(t,x(t))| \\
\leq & \frac{1}{2}\sum_{i=1}^{m}\left[(a_1+b_1\|x\|_{PC})_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}}1(t_i) + M_1\right] \\
& + \frac{1}{2}\sum_{i=1}^{m}(T-t_i)\left\{(a_1+b_1\|x\|_{PC})_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1}1(t_i) + M_2\right\} \\
& + \frac{1}{2}(a_1+b_1\|x\|_{PC})_{t_m}I_{q_m}^{\alpha_m}1(T) \\
& + \frac{T}{2}\left[\frac{1}{2}\sum_{i=1}^{m}\left\{(a_1+b_1\|x\|_{PC})_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1}1(t_i) + M_2\right\}\right. \\
& \left. + \frac{1}{2}(a_1+b_1\|x\|_{PC})_{t_m}I_{q_m}^{\alpha_m-1}1(T)\right] \\
& + \sum_{i=1}^{m}\left[(a_1+b_1\|x\|_{PC})_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}}1(t_i) + M_1\right] \\
& + \sum_{i=1}^{m}(T-t_i)\left\{(a_1+b_1\|x\|_{PC})_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1}1(t_i) + M_2\right\} \\
& + (a_1+b_1\|x\|_{PC})_{t_m}I_{q_m}^{\alpha_m}1(T) \\
= & \frac{3}{2}\sum_{i=1}^{m+1}\frac{(t_i-t_{i-1})^{\alpha_{i-1}}}{\Gamma_{q_{i-1}}(\alpha_{i-1}+1)}(a_1+b_1\|x\|_{PC}) + \frac{3}{2}mM_1 \\
& + \frac{3}{2}\sum_{i=1}^{m}(T-t_i)\left\{\frac{(t_i-t_{i-1})^{\alpha_{i-1}-1}}{\Gamma_{q_{i-1}}(\alpha_{i-1})}(a_1+b_1\|x\|_{PC}) + M_2\right\} \\
& + \frac{T}{4}\sum_{i=1}^{m+1}\frac{(t_i-t_{i-1})^{\alpha_{i-1}-1}}{\Gamma_{q_{i-1}}(\alpha_{i-1})}(a_1+b_1\|x\|_{PC}) + \frac{T}{4}M_2m \\
= & a_1\Omega_1 + \Omega_2 + b_1\|x\|_{PC}\Omega_1 \leq R,
\end{aligned}$$

which implies $\|\mathcal{A}x\|_{PC} \leq R$. Therefore, $\mathcal{A} : B_R \to B_R$.

Step 2. *The operator* $\mathcal{A} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ *is completely continuous on* B_R.

Let $\sup_{(t,x)\in J\times B_R} |f(t, x)| = F_1$. For any $\tau_1, \tau_2 \in J_k$, $k = 0, 1, \ldots, m$, with $\tau_1 < \tau_2$, we have

$$\begin{aligned} |\mathcal{A}x(\tau_2) - \mathcal{A}x(\tau_1)| \ &\leq \ |\tau_2 - \tau_1| \left[\frac{1}{2}\sum_{i=1}^{m+1} \frac{(t_i - t_{i-1})^{\alpha_{i-1}-1}}{\Gamma_{q_{i-1}}(\alpha_{i-1})} F_1 + \frac{mM_2}{2}\right] \\ &\quad + |\tau_2 - \tau_1| \sum_{i=1}^{k} \left[\frac{(t_i - t_{i-1})^{\alpha_{i-1}-1}}{\Gamma_{q_{i-1}}(\alpha_{i-1})} F_1 + M_2\right] \\ &\quad + \frac{F_1}{\Gamma_{q_k}(\alpha_k)} \left| \int_{t_k}^{\tau_2} {}_{t_k}(\tau_2 - {}_{t_k}\Phi_{q_k})_{q_k}^{(\alpha_k-1)} {}_{t_k}d_{q_k}s \right. \\ &\quad \left. - \int_{t_k}^{\tau_1} {}_{t_k}(\tau_1 - {}_{t_k}\Phi_{q_k})_{q_k}^{(\alpha_k-1)} {}_{t_k}d_{q_k}s \right|, \end{aligned}$$

which is independent of x and tends to zero as $\tau_2 - \tau_1 \to 0$. Therefore $\mathcal{A}$ is equicontinuous. Thus $\mathcal{A}\mathcal{B}_\mathcal{R}$ is relatively compact as $\mathcal{A}B_R \subset B_R$ is uniformly bounded. In view of the continuity of f, φ_k and φ_k^*, $k = 1, 2, \ldots, m$, it is clear that the operator $\mathcal{A}$ is continuous. Hence the operator $\mathcal{A} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is completely continuous on B_R. Applying the Schauder fixed point theorem, we deduce that the operator $\mathcal{A}$ has at least one fixed point in B_R. This shows that the problem (1.5) has at least one solution on J. □

In the next existence result, we make use of Leray-Schauder's nonlinear alternative. In the sequel, we set

$$\Omega_3 = \frac{3}{2}\sum_{i=1}^{m} \frac{(t_i - t_{i-1})^{\beta_{i-1}}}{\Gamma_{q_{i-1}}(\beta_{i-1} + 1)}, \tag{4.24}$$

$$\Omega_4 = \frac{3}{2}\sum_{i=1}^{m} \frac{(T - t_i)(t_i - t_{i-1})^{\gamma_{i-1}}}{\Gamma_{q_{i-1}}(\gamma_{i-1} + 1)} + \frac{T}{4}\sum_{i=1}^{m} \frac{(t_i - t_{i-1})^{\gamma_{i-1}}}{\Gamma_{q_{i-1}}(\gamma_{i-1} + 1)}. \tag{4.25}$$

Theorem 4.2.2 *Assume that*

(H_2) *there exist a continuous nondecreasing function* $\psi : [0, \infty) \to (0, \infty)$, *a continuous function* $p : J \to \mathbb{R}^+$ *with* $p^* = \sup_{t\in J} |p(t)|$ *and constants* $M_3, M_4 > 0$ *such that*

$$|f(t, x)| \leq p(t)\psi(|x|), \quad \forall(t, x) \in J \times \mathbb{R}, \tag{4.26}$$

and

$$|\varphi_k(x)| \leq M_3|x|, \quad |\varphi_k^*(x)| \leq M_4|x|, \quad \forall x \in R,\ k = 1, \ldots, m; \tag{4.27}$$

(H_3) *there exists a constant* $N > 0$ *such that*

$$\frac{(1 - M_3\Omega_3 - M_4\Omega_4)N}{p^*\psi(N)\Omega_1} > 1, \quad M_3\Omega_3 + M_4\Omega_4 < 1, \tag{4.28}$$

where Ω_3, Ω_4 *are respectively given by* (4.24) *and* (4.25).

Then the problem (1.5) *has at least one solution* J.

Proof. We shall show that the operator $\mathcal{A}$ defined by (3.1.2) has a fixed point. To accomplish this, for a positive number ρ, let $B_\rho = \{x \in PC(J,\mathbb{R}) : \|x\|_{PC} \leq \rho\}$ denote a closed ball in $PC(J,\mathbb{R})$. Then for $x \in B_\rho$, $t \in J$ and using (2.7), we have

$$\begin{aligned}
|\mathcal{A}x(t)| \leq\; & \frac{1}{2}\sum_{i=1}^{m}\left[p^*\psi(\rho)_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}}1(t_i) + \rho M_{3t_{i-1}}I_{q_{i-1}}^{\beta_{i-1}}1(t_i)\right] \\
& + \frac{1}{2}\sum_{i=1}^{m}(T - t_i)\left\{p^*\psi(\rho)_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1}1(t_i) + \rho M_{4t_{i-1}}I_{q_{i-1}}^{\gamma_{i-1}}1(t_i)\right\} \\
& + \frac{1}{2}p^*\psi(\rho)_{t_m}I_{q_m}^{\alpha_m}1(T) + \frac{T}{2}\left[\frac{1}{2}\sum_{i=1}^{m}\left\{p^*\psi(\rho)_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1}1(t_i)\right.\right. \\
& \left.\left. + \rho M_{4t_{i-1}}I_{q_{i-1}}^{\gamma_{i-1}}1(t_i)\right\} + \frac{1}{2}p^*\psi(\rho)_{t_m}I_{q_m}^{\alpha_m-1}1(T)\right] \\
& + \sum_{i=1}^{m}\left[p^*\psi(\rho)_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}}1(t_i) + \rho M_{3t_{i-1}}I_{q_{i-1}}^{\beta_{i-1}}1(t_i)\right] \\
& + \sum_{i=1}^{m}(T - t_i)\left\{p^*\psi(\rho)_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1}1(t_i) + \rho M_{4t_{i-1}}I_{q_{i-1}}^{\gamma_{i-1}}1(t_i)\right\} \\
& + p^*\psi(\rho)_{t_m}I_{q_m}^{\alpha_m}1(T) \\
=\; & p^*\psi(\rho)\Omega_1 + \rho M_3\Omega_3 + \rho M_4\Omega_4 := K,
\end{aligned}$$

which implies that $\|\mathcal{A}x\|_{PC} \leq K$.

To show that the operator $\mathcal{A}$ maps bounded sets into equicontinuous sets of $PC(J,\mathbb{R})$, we take $\tau_1, \tau_2 \in J_k$ for some $k \in \{0, 1, 2, \ldots, m\}$ with $\tau_1 < \tau_2$ and

$x \in B_\rho$. Then we have

$$
\begin{aligned}
& |\mathcal{A}x(\tau_2) - \mathcal{A}x(\tau_1)| \\
\leq\ & |\tau_2 - \tau_1| \left[\frac{p^*\psi(\rho)}{2} \sum_{i=1}^{m+1} \frac{(t_i - t_{i-1})^{\alpha_{i-1}-1}}{\Gamma_{q_{i-1}}(\alpha_{i-1})} + \frac{\rho M_4}{2} \sum_{i=1}^{m} \frac{(t_i - t_{i-1})^{\gamma_{i-1}}}{\Gamma_{q_{i-1}}(\gamma_{i-1}+1)} \right] \\
& + |\tau_2 - \tau_1| \sum_{i=1}^{k} \left[\frac{(t_i - t_{i-1})^{\alpha_{i-1}-1}}{\Gamma_{q_{i-1}}(\alpha_{i-1})} p^*\psi(\rho) + \rho M_4 \frac{(t_i - t_{i-1})^{\gamma_{i-1}}}{\Gamma_{q_{i-1}}(\gamma_{i-1}+1)} \right] \\
& + \frac{p^*\psi(\rho)}{\Gamma_{q_k}(\alpha_k)} \left| \int_{t_k}^{\tau_2} {}_{t_k}(\tau_2 - {}_{t_k}\Phi_{q_k})_{q_k}^{(\alpha_k - 1)}\, {}_{t_k}d_{q_k}s \right. \\
& \left. - \int_{t_k}^{\tau_1} {}_{t_k}(\tau_1 - {}_{t_k}\Phi_{q_k})_{q_k}^{(\alpha_k - 1)}\, {}_{t_k}d_{q_k}s \right|,
\end{aligned}
$$

which tends to zero independent of x as $\tau_1 \to \tau_2$. Thus, by Arzelá-Ascoli theorem, the operator $\mathcal{A} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is completely continuous.

Finally, for $\lambda \in (0, 1)$, let $x = \lambda \mathcal{A}x$. Then, as in the first step, we can get

$$\|x\|_{PC} \leq p^*\psi(\|x\|_{PC})\Omega_1 + \|x\|_{PC} M_3\Omega_3 + \|x\|_{PC} M_4\Omega_4,$$

which can alternatively be written as

$$\frac{(1 - M_3\Omega_3 - M_4\Omega_4)\|x\|_{PC}}{p^*\psi(\|x\|_{PC})\Omega_1} \leq 1.$$

In view of (H_3), there exists N such that $\|x\|_{PC} \neq N$. We define $\mathcal{U} = \{x \in PC(J, \mathbb{R}) : \|x\|_{PC} < N\}$. Note that the operator $\mathcal{A} : \overline{\mathcal{U}} \to PC$ is continuous and completely continuous. From the choice of $\mathcal{U}$, there is no $x \in \partial\mathcal{U}$ such that $x = \lambda\mathcal{A}x$ for some $\lambda \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 2.4.17), we deduce that $\mathcal{A}$ has a fixed point $x \in \overline{\mathcal{U}}$ which is a solution of the problem (1.5) on J. This completes the proof. □

In the last theorem, we apply Banach's contraction principle to establish the uniqueness of solutions for the problem (1.4).

Theorem 4.2.3 *Assume that there exist a function $\mathcal{W}(t) \in C(J, \mathbb{R}^+)$ with $W = \sup_{t \in J} |\mathcal{W}(t)|$ and positive constants M_5, M_6 such that*

$$|f(t, x) - f(t, y)| \leq \mathcal{W}(t)|x - y|, \quad \forall (t, x) \in J \times \mathbb{R}, \tag{4.29}$$

and

$$|\varphi_k(x) - \varphi_k(y)| \leq M_5|x - y|, \ |\varphi_k^*(x) - \varphi_k^*(y)| \leq M_6|x - y|, \ x, y \in \mathbb{R}, \tag{4.30}$$

for $k = 1, 2, \ldots, m$. If

$$W\Omega_1 + M_5\Omega_3 + M_6\Omega_4 < 1, \tag{4.31}$$

then the problem (1.5) has a unique solution on J.

Proof. For any $x, y \in PC(J, \mathbb{R})$, we have

$$\begin{aligned}
& |\mathcal{A}x(t) - \mathcal{A}y(t)| \\
\leq\ & \frac{1}{2}\sum_{i=1}^{m}\left[{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}}|f(t_i,x) - f(t_i,y)| + M_5\,{}_{t_{i-1}}I_{q_{i-1}}^{\beta_{i-1}}|x-y|(t_i)\right] \\
& + \frac{1}{2}\sum_{i=1}^{m}(T-t_i)\left\{{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1}|f(t_i,x) - f(t_i,y)| + M_6\,{}_{t_{i-1}}I_{q_{i-1}}^{\gamma_{i-1}}|x-y|(t_i)\right\} \\
& + \frac{1}{2}\,{}_{t_m}I_{q_m}^{\alpha_m}|f(T,x) - f(T,y)| + \frac{T}{2}\left[\frac{1}{2}\sum_{i=1}^{m}\left\{{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1}|f(t_i,x) - f(t_i,y)|\right.\right. \\
& \left.\left. + M_6\,{}_{t_{i-1}}I_{q_{i-1}}^{\gamma_{i-1}}|x-y|(t_i)\right\} + \frac{1}{2}\,{}_{t_m}I_{q_m}^{\alpha_m-1}|f(T,x) - f(T,y)|\right] \\
& + \sum_{i=1}^{k}\left[{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}}|f(t_i,x) - f(t_i,y)| + M_5\,{}_{t_{i-1}}I_{q_{i-1}}^{\beta_{i-1}}|x-y|(t_i)\right] \\
& + \sum_{i=1}^{k}(t-t_i)\left\{{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}-1}|f(t_i,x) - f(t_i,y)| + M_6\,{}_{t_{i-1}}I_{q_{i-1}}^{\gamma_{i-1}}|x-y|(t_i)\right\} \\
& + {}_{t_k}I_{q_k}^{\alpha_k}|f(t,x) - f(t,y)| \\
\leq\ & (W\Omega_1 + M_5\Omega_3 + M_6\Omega_4)\|x-y\|_{PC},
\end{aligned}$$

which yields

$$\|\mathcal{A}x - \mathcal{A}y\|_{PC} \leq (W\Omega_1 + M_5\Omega_3 + M_6\Omega_4)\|x-y\|_{PC}.$$

By (4.31), we conclude that $\mathcal{A}$ is a contraction. Thus, by Banach's contraction mapping principle, the problem (1.5) has a unique solution on J. This completes the proof. □

Chapter 5

Conclusions

In this chapter, we give conclusions by presenting some examples for the illustration of our results.

5.1 Examples of BVP. (1.4)

Example 5.1.1 *Consider the following boundary value problem of impulsive fractional q-integro-difference equations with separated boundary conditions:*

$$\begin{cases} {}^{c}_{t_k}D^{\frac{2k^2+3}{k^2+2}}_{\frac{3k+2}{5k+3}}x(t) = \dfrac{2t|x(t)|}{3+|x(t)|}e^{-x^2(t)}\cos\left({}_{t_k}I^{\frac{2k+1}{2}}_{\frac{3k+2}{5k+3}}x(t)\right)^2 + \dfrac{1}{2}, \quad t \in [0,4/3]\setminus\{t_1,t_2,t_3\}, \\ \Delta x(t_k) = \dfrac{k}{66\pi(k+1)}\sin(\pi x(t_k)), \quad t_k = \dfrac{k}{3}, \ k=1,2,3, \\ {}_{t_k}D_{\frac{3k+2}{5k+3}}x(t_k^+) - {}^{c}_{t_{k-1}}D^{\frac{k}{k^2-2k+3}}_{\frac{3k-1}{5k-2}}x(t_k) = \dfrac{|x(t_k)|}{78k(1+|x(t)|)}, \ t_k = \dfrac{k}{2}, \\ \dfrac{1}{2}x(0) + \dfrac{2}{3}\,{}_0D_{\frac{2}{3}}x(0) = 0, \qquad \dfrac{2}{5}x\left(\dfrac{4}{3}\right) + \dfrac{3}{4}\,{}^{c}_{1}D^{\frac{4}{11}}_{\frac{11}{18}}x\left(\dfrac{4}{3}\right) = 0. \end{cases} \tag{5.1}$$

Here $\alpha_k = (2k^2+3)/(k^2+2)$, $q_k = (3k+2)/(5k+3)$, $\beta_k = (2k+1)/2$, $\gamma_k = (k+1)/(k^2+2)$, $k = 0,1,2,3$, $t_k = k/3$, $k = 1,2,3$, $m = 3$, $T = 4/3$, $\lambda_1 = 1/2$, $\lambda_2 = 2/3$, $\xi_1 = 2/5$ and $\xi_2 = 3/4$. With the given information, it is found that $|\Delta| = 0.0432973538$. Also, we have

$$\begin{aligned} |f(t,x,y)| &= \left|\frac{2t|x|}{3+|x|}e^{-x^2}\cos(y)^2 + \frac{1}{2}\right| \le 2t + \frac{1}{2}, \\ |\varphi_k(x)| &= \left|\frac{k}{66\pi(k+1)}\sin(\pi x)\right| \le \frac{1}{66\pi}, \ |\varphi_k^*(x)| = \left|\frac{|x|}{78k(1+|x|)}\right| \le \frac{1}{78}, \ k=1,2,3. \end{aligned}$$

With $L_1 = 1/66$ and $L_2 = 1/78$, we obtain $L\Lambda_2 = 0.9927769903 < 1$. Thus all the conditions of Theorem 4.1.1 are satisfied. Therefore, by the conclusion of Theorem 4.1.1, problem (5.1) has at least one solution on $[0, 4/3]$.

Example 5.1.2 *Consider the following impulsive boundary value problem*

$$\begin{cases} {}^{c}_{t_k}D^{\frac{2k^2+2k+5}{k^2+2k+3}}_{\frac{k^2+k+2}{k^2+2k+3}} x(t) = \dfrac{t^2}{100}\left(|x(t)| + \dfrac{1+|x(t)|}{2+|x(t)|}\right) + {}_{t_k}I^{\frac{k^2+2k+4}{k^2+2k+3}}_{\frac{k^2+k+2}{k^2+2k+3}} x(t), \ t \in [0,1] \setminus \{t_1, \ldots, t_4\}, \\ \Delta x(t_k) = \dfrac{1}{40k\pi} \arctan\left(\dfrac{4k\pi}{5}|x(t_k)|\right), \quad t_k = \dfrac{k}{5}, \ k = 1, \ldots, 4, \\ {}_{t_k}D_{\frac{k^2+k+2}{k^2+2k+3}} x(t_k^+) - {}_{t_{k-1}}D^{\frac{k^2-k+1}{k^2+2}}_{\frac{k^2-k+2}{k^2+2}} x(t_k) = \dfrac{\sin(k\pi/7)}{85} \dfrac{|x(t_k)|}{(1+|x(t_k)|)}, \ t_k = \dfrac{k}{5}, \\ \dfrac{1}{5}x(0) + \dfrac{3}{10}\, {}_0D_{\frac{2}{3}} x(0) = 0, \qquad \dfrac{4}{15}x(1) + \dfrac{7}{20}\, {}^{c}_{\frac{4}{5}}D^{\frac{21}{27}}_{\frac{22}{27}} x(1) = 0. \end{cases} \tag{5.2}$$

Here $\alpha_k = (2k^2 + 2k + 5)/(k^2 + 2k + 3)$, $q_k = (k^2 + k + 2)/(k^2 + 2k + 3)$, $\beta_k = (k^2+2k+4)/(k^2+2k+3)$, $\gamma_k = (k^2+k+1)/(k^2+2k+3)$, $k = 0, 1, 2, 3, 4$, $t_k = k/5$, $k = 1, 2, 3, 4$, $m = 4$, $T = 1$, $\lambda_1 = 1/5$, $\lambda_2 = 3/10$, $\xi_1 = 4/15$ and $\xi_2 = 7/20$. Using the above data, we find that $|\Delta| = 0.02280828040 \neq 0$, $\Lambda_1 = 6.560295012$, $\Lambda_3 = 0.5907970651$, $\Lambda_4 = 21.29456817$ and $\Lambda_5 = 17.30334490$. To find $\Lambda_2(N)$, we see that

$$\begin{aligned} |\varphi_k(x)| &= \left|\frac{1}{40k\pi} \arctan\left(\frac{4k\pi}{5}|x(t_k)|\right)\right| \le \frac{1}{80} := N_1 \\ |\varphi_k^*(x)| &= \left|\frac{\sin(k\pi/7)}{85} \frac{|x(t_k)|}{(1+|x(t_k)|)}\right| \le \frac{1}{85} := N_2, \end{aligned}$$

for $k = 1, 2, 3, 4$, which leads to $\Lambda_2(N) = 0.4697508657$ and also (H_3) is satisfied. Choosing two continuous nondecreasing functions $\omega_1, \omega_2 : [0, \infty) \to [0, \infty)$ as

$$\omega_1(x) = \frac{1}{50}x, \qquad \omega_2(x) = \frac{1}{85}x,$$

we obtain

$$\begin{aligned} |\varphi_k(x) - \varphi_k(y)| &\le \frac{1}{50}|x-y| = \omega_1(|x-y|), \\ |\varphi_k^*(x) - \varphi_k^*(y)| &\le \frac{1}{85}|x-y| = \omega_2(|x-y|). \end{aligned}$$

Setting $D_1 = 1/50$ and $D_2 = 1/85$, it follows that $D_1\Lambda_4 + D_2\Lambda_5 = 0.6294601269 < 1$. Thus the condition (H_5) holds. In addition, we have

$$\begin{aligned} |f(y, x, y)| &= \left|\frac{t^2}{100}\left(|x| + \frac{1+|x|}{2+|x|}\right) + y\right| \\ &\le \frac{t^2}{200}(x^2 + 3|x| + 1) + |y|. \end{aligned}$$

Thus (H_4) is satisfied with $p(t)=t^2/200$ and $\psi(|x|)=x^2+3|x|+1$. Further,

$$\sup_{r\in(0,\infty)}\frac{r}{p^*\psi(r)\Lambda_1+\Lambda_2(N)}=2.815409997>\frac{1}{1-\Lambda_3}=2.443775239,$$

where $p^*=1/200$. Therefore the condition (H_6) holds. Thus the conclusion of Theorem 4.1.2 implies that the problem (5.2) has at least one solution on $[0,1]$.

Example 5.1.3 *Consider the problem*

$$\begin{cases} {}^{c}_{t_k}D^{\frac{4k^2+4k+3}{3k^2+2k+2}}_{\frac{2k^2+k+3}{2k^2+3k+4}}x(t)=\dfrac{\sin^2 t}{2(t^2+30)}\left(\dfrac{x^2(t)+2|x(t)|}{1+|x(t)|}\right)+\dfrac{3e^{-t}}{t+10}\,{}_{t_k}I^{\frac{3k^2+2k+3}{3k^2+2k+2}}_{\frac{2k^2+k+3}{2k^2+3k+4}}x(t)+\dfrac{4}{3}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t\in[0,5/4]\setminus\{t_1,\ldots,t_4\}, \\ \Delta x(t_k)=\dfrac{1}{100k}\left(\dfrac{x^2(t_k)+2|x(t_k)|}{1+|x(t_k)|}\right)+\dfrac{3}{2},\quad t_k=\dfrac{k}{4},\ k=1,\ldots,4, \\ {}_{t_k}D_{\frac{2k^2+k+3}{2k^2+3k+4}}x(t_k^+)-{}^{c}_{t_{k-1}}D^{\frac{2k^2-2k+1}{3k^2-4k+3}}_{\frac{2k^2-3k+4}{2k^2-k+3}}x(t_k)=\dfrac{1}{50k}\sin(x(t_k))+\dfrac{2}{5},\ t_k=\dfrac{k}{4}, \\ \dfrac{2}{5}x(0)+\dfrac{3}{7}\,{}_0D_{\frac{3}{4}}x(0)=0,\qquad \dfrac{4}{9}x\left(\dfrac{5}{4}\right)+\dfrac{5}{8}\,{}^{c}_{1}D^{\frac{41}{58}}_{\frac{39}{48}}x\left(\dfrac{5}{4}\right)=0. \end{cases} \tag{5.3}$$

Here $\alpha_k=(4k^2+4k+3)/(3k^2+2k+2)$, $q_k=(2k^2+k+3)/(2k^2+3k+4)$, $\beta_k=(3k^2+2k+3)/(3k^2+2k+2)$, $\gamma_k=(2k^2+2k+1)/(3k^2+2k+2)$, $k=0,1,2,3,4$, $t_k=k/4$, $k=1,2,3,4$, $m=4$, $T=5/4$, $\lambda_1=2/5$, $\lambda_2=3/7$, $\xi_1=4/9$ and $\xi_2=5/8$. With the given values, we find that $|\Delta|=0.1515229972\neq 0$, $\Lambda_1=6.322759092$, $\Lambda_3=0.7537203342$, $\Lambda_4=11.94663857$ and $\Lambda_5=11.27343204$. Also, we have

$$\begin{aligned} |f(t,x_1,y_1)-f(t,x_2,y_2)| &\le \left|\frac{\sin^2 t}{(t^2+30)}\right||x_1-x_2|+\left|\frac{3e^{-t}}{t+10}\right||y_1-y_2|, \\ |\varphi_k(x)-\varphi_k(y)| &\le \frac{1}{50}|x-y|, \\ |\varphi_k^*(x)-\varphi_k^*(y)| &\le \frac{1}{50}|x-y|,\quad k=1,2,3,4. \end{aligned}$$

Clearly $M_1^*=1/30$ and $M_2^*=3/10$. Hence, $M_1^*\Lambda_1+M_2^*\Lambda_3+\Lambda_4L_1+\Lambda_5L_2=0.9012761489<1$. Thus all the conditions of Theorem 4.1.3 are satisfied. Therefore, it follows by the conclusion of Theorem 4.1.3 that problem (5.3) has a unique solution on $[0,5/4]$.

5.2 Examples of BVP. (1.5)

Example 5.2.1 *Consider the following anti-periodic boundary value problem for impulsive Caputo fractional q-difference equations:*

$$\begin{cases} {}^c_{t_k}D^{\frac{k+3}{k+2}}_{\frac{1}{k^2-3k+4}}x(t) = 2t^2 + 1 + \dfrac{1}{8}\sin^2 t \dfrac{x^2(t)}{1+|x(t)|}, & t \in [0,2] \setminus \{t_1, t_2, t_3\}, \\ \Delta x(t_k) = \dfrac{k}{k+1} e^{1-\left({}_{t_{k-1}}I^{\frac{2k-1}{2}}_{\frac{1}{k^2-5k+8}} x(t_k)\right)^2}, & t_k = \dfrac{k}{2},\ k = 1,2,3, \\ {}_{t_k}D_{\frac{1}{k^2-3k+4}} x(t_k^+) - {}_{t_{k-1}}D_{\frac{1}{k^2-5k+8}} x(t_k) = k^2 \cos\left(\log\left(1 + \left|{}_{t_{k-1}}I^{\frac{2k+5}{2}}_{\frac{1}{k^2-5k+8}} x(t_k)\right|\right)\right), & t_k = \dfrac{k}{2}, \\ x(0) = -x(2), \qquad {}_0D_{\frac{1}{4}}x(0) = -{}_{\frac{3}{2}}D_{\frac{1}{4}}x(2). \end{cases} \tag{5.4}$$

Here $\alpha_k = (k+3)/(k+2)$, $q_k = 1/(k^2-3k+4)$, $k = 0,1,2,3$, $\beta_{k-1} = (2k-1)/2$, $\gamma_{k-1} = (2k+5)/2$, $t_k = k/2$, $k = 1,2,3$, $m = 3$, $T = 2$. With the given information, it is found that $\Omega_1 = 7.575532753$. Also, we have

$$|f(t,x)| = \left|2t^2 + 1 + \frac{1}{8}\sin^2 t \frac{x^2}{1+|x|}\right| \le 2t^2 + 1 + \frac{1}{8}\sin^2 t|x|,$$

$$|\varphi_k(y)| = \left|\frac{k}{k+1}e^{1-y^2}\right| \le e, \ |\varphi_k^*(z)| = \left|k^2\cos\left(\log\left(1+|z|\right)\right)\right| \le 9, \ k = 1,2,3.$$

With $B = \sup_{t\in[0,2]} |(1/8)\sin^2 t| = 1/8$, we obtain $B\Omega_1 = 0.9469415941 < 1$. Thus all the conditions of Theorem 4.2.1 are satisfied. Therefore, by the conclusion of Theorem 4.2.1, the problem (5.4) has at least one solution on $[0,2]$.

Example 5.2.2 *Consider the anti-periodic impulsive boundary value problem of fractional q-difference equations given by*

$$\begin{cases} {}^c_{t_k}D^{\frac{k^2+5}{k^2+3}}_{\frac{1}{k^2-4k+6}}x(t) = \dfrac{1}{(2+t)^2}\left(\log_e\left(\dfrac{|x(t)|}{4}+2\right)\right)^2, & t \in [0,5/3] \setminus \{t_1,\ldots,t_4\}, \\ \Delta x(t_k) = \dfrac{1}{11+k}\sin\left({}_{t_{k-1}}I^{\frac{2+(-1)^{k-1}}{2}}_{\frac{1}{k^2-6k+11}} x(t_k)\right), & t_k = \dfrac{k}{3},\ k = 1,\ldots,4, \\ {}_{t_k}D_{\frac{1}{k^2-4k+6}} x(t_k^+) - {}_{t_{k-1}}D_{\frac{1}{k^2-6k+11}} x(t_k) = \dfrac{1}{3+k}\, {}_{t_{k-1}}I^{\frac{4+(-1)^{k-1}}{2}}_{\frac{1}{k^2-6k+11}} x(t_k), & t_k = \dfrac{k}{3}, \\ x(0) = -x\left(\dfrac{5}{3}\right), \qquad {}_0D_{\frac{1}{6}}x(0) = -{}_{\frac{4}{3}}D_{\frac{1}{6}}x\left(\dfrac{5}{3}\right). \end{cases} \tag{5.5}$$

Here $\alpha_k = (k^2+5)/(k^2+3)$, $q_k = 1/(k^2-4k+6)$, $k = 0,1,2,3,4$, $\beta_{k-1} = (2+(-1)^{k-1})/2$, $\gamma_{k-1} = (4+(-1)^{k-1})/2$, $t_k = k/3$, $k = 1,2,3,4$, $m = 4$, $T = 5/3$. Using the above data, we find that $\Omega_1 = 6.316994013$, $\Omega_3 = 2.358729544$, $\Omega_4 = 0.6481929403$, and

$$|f(t,x)| = \left|\frac{1}{(2+t)^2}\left(\log_e\left(\frac{|x|}{4}+2\right)\right)^2\right| \leq \frac{1}{(2+t)^2}\left(\frac{|x|}{4}+2\right),$$

$$|\varphi_k(y)| = \frac{1}{11+k}|\sin y| \leq \frac{1}{12}|y|,\ |\varphi_k^*(z)| = \frac{1}{3+k}|z| \leq \frac{1}{4}|z|,\ k = 1,2,3,4.$$

Setting $\psi(x) = (x/4)+2$, $p^* = \sup_{t\in[0,5/3]}|1/(2+t)^2|=1/4$, $M_3 = 1/12$ and $M_4 = 1/4$, we find that $M_3\Omega_3 + M_4\Omega_4 = 0.3586090304 < 1$. Also, there exists a constant N such that $N > 12.80927819$ satisfying (4.28). Clearly the hypothesis of Theorem 4.2.2 holds true. Thus the conclusion of Theorem 4.2.2 implies that the problem (5.5) has at least one solution on $[0,5/3]$.

Example 5.2.3 *Consider the following impulsive anti-periodic problem of fractional q-difference equation:*

$$\begin{cases} {}^{c}_{t_k}D^{\frac{k^2+k+3}{k^2+2}}_{\frac{1}{k^2-5k+8}}x(t) = \dfrac{e^{-t}\sin(2t+1)}{t^2+30}\dfrac{x^2(t)+2|x(t)|}{1+|x(t)|} + \dfrac{1}{2}, \quad t \in [0,3/2]\setminus\{t_1,\ldots,t_5\}, \\ \Delta x(t_k) = \dfrac{7k}{25}\tan^{-1}\left({}_{t_{k-1}}I^{\frac{2k+1}{2}}_{\frac{1}{k^2-7k+14}}x(t_k)\right) + \dfrac{2}{3}, \quad t_k = \dfrac{k}{4},\ k = 1,\ldots,5, \\ {}_{t_k}D_{\frac{1}{k^2-5k+8}}x(t_k^+) - {}_{t_{k-1}}D_{\frac{1}{k^2-7k+14}}x(t_k) = \dfrac{\left|{}_{t_{k-1}}I^{\frac{2k^2-4k+3}{2}}_{\frac{1}{k^2-7k+14}}x(t_k)\right|}{5k\left(1+\left|{}_{t_{k-1}}I^{\frac{2k^2-4k+3}{2}}_{\frac{1}{k^2-7k+14}}x(t_k)\right|\right)} + \dfrac{3}{4},\ t_k = \dfrac{k}{4}, \\ x(0) = -x\left(\dfrac{3}{2}\right), \qquad {}_0D_{\frac{1}{8}}x(0) = -{}_{\frac{5}{4}}D_{\frac{1}{8}}x\left(\dfrac{3}{2}\right). \end{cases} \tag{5.6}$$

Here $\alpha_k = (k^2+k+3)/(k^2+2)$, $q_k = 1/(k^2-5k+8)$, $k = 0,1,2,3,4,5$, $\beta_{k-1} = (2k+1)/2$, $\gamma_{k-1} = (2k^2-4k+3)/2$, $t_k = k/4$, $k = 1,2,3,4,5$, $m = 5$, $T = 3/2$. With the given values, we find that $\Omega_1 = 5.173430458$, $\Omega_3 = 0.2141916028$ and $\Omega_4 = 1.375385103$. Also, we have

$$|f(t,x_1) - f(t,x_2)| \leq \left|\frac{2e^{-t}\sin(2t+1)}{t^2+30}\right||x_1 - x_2|,$$

$$|\varphi_k(y_1) - \varphi_k(y_2)| = \frac{7k}{25}\left|\tan^{-1}y_1 - \tan^{-1}y_2\right| \leq \frac{7}{5}|y_1 - y_2|,$$

$$|\varphi_k^*(z_1) - \varphi_k^*(z_2)| = \frac{1}{5k}\left|\frac{|z_1|}{1+|z_1|} - \frac{|z_2|}{1+|z_2|}\right| \leq \frac{1}{5}|z_1 - z_2|,\ k = 1,2,3,4,5.$$

It is easy to see that $W = 1/15$. Hence, $W\Omega_1 + M_5\Omega_3 + M_6\Omega_4 = 0.9198406284 < 1$. Thus all the conditions of Theorem 4.1.3 are satisfied. Hence it follows by the conclusion of Theorem 4.2.3 that the problem (5.6) has a unique solution on $[0, 3/2]$.

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