The new modified Jarratt method for the approximate solution of different type of differential equations

By

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วิธีการของจารัตต์ปรับปรุงใหม่สำหรับการประมาณคำตอบของสมการเชิงอนุพันธ์ชนิดต่างๆ

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Abstract

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The purpose of this project is to introduce the new modified Jarratt method to find the approximate solution of an ordinary differential equation with an initial condition. Also, some numerical examples with initial conditions are given to show the properties of the iteration method. The results of absolute errors are compared with Newton, Euler, Runge-Kutta and Picard iteration methods. Finally, the presented method, namely the new modified Jarratt method, is highlighted as one of the effective and efficient ways in solving different type of the problem.

Keywords : Newton's methods, Jarratt method; Banach spaces

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Contents

CHAPTER I

INTRODUCTION

Solving nonlinear equations is one of the most important problems in numerical analysis. In this study, the iteration methods are considered for finding a solution x^* of a nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where $F : I \subseteq R \rightarrow R$ is a scalar function for an open interval I.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $x = Q(x)$ for some suitable operator Q, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems.

The unknowns of engineering equations can be functions (difference, differential and integral equations), vectors (systems of linear or nonlinear algebraic equations) or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative when starting from one or several initial approximations, a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework. Many authors have developed high order methods for generating a sequence approx- imating x^*. A survey of such results can be found in [1, and the references there] (see also [2-11]).

The natural generalization of the Newton method is to apply a multipoint scheme. Suppose that we know the analytic expressions of $F(x_n), F'(x_n)$ and $F'(x_n)^{-1}$ at a recurrent step x_n for each $n \geq 0$. In order to increase the order of convergence and to avoid the computation of the second Frechet-derivative, we can add one more evaluation of $F(c_1x_n + c_2y_n)$ or $F'(c_1x_n + c_2y_n)$, where c_1 and c_2 are real constants that are independent of x_n and y_n, whereas y_n is generated by a Newton-step. A two-point scheme for functions of one variable was found and developed by Ostrowski [11].

CHAPTER II

PRELIMINARIES

In this chapter, we give some definitions, notations, and some useful results that will be used in the later chapters.

Definition 2.1. local convergence, assume x^* exists, it is shown that there is a neighborhood about x^* such that the iterates converge to x^*

Definition 2.2. semilocal convergence, for a particular choice of initial values, the iterates converge to a solution x^*

Definition 2.3. Let $L \leq 0, M_0 > 0, M > 0, N \leq 0 and \eta > 0$ be given constants. Define the polynomials on $[0, +\infty)$ for some $\alpha > 0$ by

$$f_1(t) = (1 + M\eta)M\eta t + 4M_0\alpha(\alpha + 2)\eta - 2\alpha,$$

$$g(t) = M(1 + M\eta)Mt^2 + [4M_0\alpha(1 + \alpha)] - M(1 + M\eta)Mt - 4M_0\alpha,$$

$$h_1(t) = M_0\eta(1 + \alpha)t^2 + M_0\eta(1 + \alpha)t + \frac{M\alpha^2\eta}{2} + \frac{13L\eta^3}{108} + \frac{2N\alpha\eta}{9M} + \frac{2\alpha M\eta}{3} - 1,$$

$$g_1(t) = M_0\eta(1 + \alpha)t^2 + \left(\frac{M\alpha^2}{2} + \frac{13L\eta^2}{108} + \frac{2N\alpha}{9M} + \frac{2\alpha M}{3} - M_0\eta\right)t - \left(\frac{M\alpha^2}{2} + \frac{13L\eta^2}{108} + \frac{2N\alpha}{9M} + \frac{2\alpha M}{3} - M_0\eta\right).$$

Moreover, define a scalar ϕ_0 by

$$\phi_0 = \frac{\left[\frac{M\alpha^2}{2} + \frac{13L\eta^3}{108} + \frac{2N\alpha\eta}{9M} + \frac{2\alpha M\eta}{3}\right]}{1 - M_0[\eta + \frac{M(1+M\eta)}{2}\eta^2]}.$$

The polynomials f_1, g, g_1 have unique positive roots denoted by ϕ_{f_1}, ϕ_g and ϕ_{g_1} (given in an explicit form), respectively, by the Descartes rule of signs. Moreover, assume

$$M_0\left[\eta + \frac{M(1 + M\eta)}{2}\eta^2\right] < 1 \tag{2.2}$$

and

$$\frac{M\alpha^2\eta}{2} + \frac{13L\eta^3}{108} + \frac{2N\alpha\eta}{9M} + \frac{2\alpha M\eta}{3} < 1 \tag{2.3}$$

Under the conditions (2.2), (5.7), respectively,

$$\phi_0 > 0,$$

and the polynomial h_1 has a unique positive root ϕ_{h_1}.

Set $\phi_1 = \min\{\phi_{h_1}, \phi_{f_1}, \phi_g, \phi_{g_1}, 1\}$. Furthermore, assume

$$\phi_0 > \phi_1. \tag{2.4}$$

If $\phi_1 = 1$, then assume that (2.4) holds as a strict inequality. From now on (2.2) − (2.4) constitute the (C) conditions.

Theorem 2.4. *[6] Let $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be thrice differentiable. Assume that there exist $x_0 \in \mathcal{D}, L \geq 0, M \geq 0, N \geq 0$ and $\eta \geq 0$ such that*

$$F^{-1}(x_0) \in \mathcal{L}(\mathcal{Y}, \mathcal{X}), \tag{2.5}$$

$$\|F^{-1}(x_0)F(x_0)\| \leq \eta, \tag{2.6}$$

$$\|F^{-1}(x_0)F''(x_0)\| \leq M, \tag{2.7}$$

$$\|F^{-1}(x_0)F'''(x_0)\| \leq N, \tag{2.8}$$

$$\|F^{-1}(x_0)(F'''(x) - F'''(y))\| \leq L\|x - y\|, \tag{2.9}$$

for each $x, y \in \mathcal{D}$,

$$M\left(1 + \frac{N}{6M^2} + \frac{13L}{36M^2}\right)^{\frac{1}{3}} \leq K, \tag{2.10}$$

$$h = K\eta \leq 0.46568 \tag{2.11}$$

and

$$\overline{U}(x_0, v^*) = \{x \in X, \|x - x_0\| \leq v^*\} \subseteq \mathcal{D}, \tag{2.12}$$

where v^* and v^{**} are the zeros of functions

$$g(t) = \frac{K}{2}t^2 - t + \eta \tag{2.13}$$

given by

$$v^* = \frac{1-\sqrt{1-2h}}{h}\eta, \quad v^{**} = \frac{1+\sqrt{1-2h}}{h}\eta, \tag{2.14}$$

Then the following hold:

(1) *The scalar sequences* $\{v_n\}$ *and* $\{w_n\}$ *given by*

$$\begin{cases} w_n = v_n - g^{-1}(v_n)g(v_n), \\ b_n = g^{-1}(v_n)(g'(v_n + \frac{2}{3}(w_n - v_n)) - g'(v_n)), \\ v_{n+1} = w_n - \frac{3}{4}b_n(1 - \frac{3}{2}b_n)(w_n - v_n) \end{cases} \tag{2.15}$$

for each $n \geq 0$ *are non-decreasing and converge to their common limit* v^*, *so that*

$$v_n \leq w_n \leq v_{n+1} \leq w_{n+1} \tag{2.16}$$

(2) *The sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by (JM) are well defined, remain in* $\overline{U}(x_0, v^*)$ *for all* $n \leq 0$ *and converge to a unique solution* $x^* \in \overline{U}(x_0, v^*)$ *of the equation* $F(x) = 0$, *which is the unique solution of the equation* $F(x) = 0 \in U(x_0, v^{**})$. *Moreover, the following estimates hold for all* $n \leq 0$:

$$\|y_n - x_n\| \leq w_n - v_n, \tag{2.17}$$

$$\|x_{n+1} - y_n\| \leq w_{n+1} - v_n, \tag{2.18}$$

$$\|y_n - x^*\| \leq v^* - w_n, \tag{2.19}$$

$$\|x_n - x^*\| \leq v^* - v_n \leq \frac{(1-\theta)^2\eta(\sqrt[3]{5}\theta)^{4^n-1}}{1 - \frac{1}{\sqrt[3]{5}}(\sqrt[3]{5}\theta)^{4^n}}, \tag{2.20}$$

where

$$\theta = \frac{v^*}{v^{**}} \tag{2.21}$$

Remark 2.5. The bounds of Theorem 2.4 can be improved under the same hypotheses and computational cost in two cases as follows.

Case 1. Define a function g_0 by

$$g_0(t) = \frac{M_0}{2}t^2 - t + \eta. \tag{2.22}$$

In view of (2.6), there exists $M_0 \in [0, M]$ such that

$$\|F^{-1}(x_0)(F'(x) - F'(x_0))\| \le M_0\|x - x_0\|, \tag{2.23}$$

for all $x \in \mathcal{D}$. We can find upper bounds on the norms $\|F^{-1}(x_0)F'(x_0)\|$ using M_0, which is actually needed, and not K used in [4]. Note that

$$M_0 \le K \tag{2.24}$$

and K/M_0 can be arbitrarily large [1-3]. Using (2.23), it follows that, for any $x \in \overline{U}(x_0, v^*)$,

$$\|F^{-1}(x_0)(F'(x) - F'(x_0))\| \le M_0\|x - x_0\| \le K\|x - x_0\| \le Kv^* < 1, \tag{2.25}$$

It follows from (2.25) and the Banach lemma on invertible operators [1] that $\|F'(x)^{-1}F'(x_0)\|$ exists and

$$\|F^{-1}(x)F'(x_0)\| \le \frac{1}{1 - M_0\|x - x_0\|}. \tag{2.26}$$

We can use (2.25) instead of the less precise one used in [4]:

$$\|F^{-1}(x)F'(x_0)\| \le \frac{1}{1 - K\|x - x_0\|}. \tag{2.27}$$

This suggests that more precise scalar majorizing sequences $\{\overline{v}_n\}, \{\overline{w}_n\}$ can be used and they are defined as follows for initial iterates $\overline{v}_n = 0, \overline{w}_1 = \eta$:

$$\begin{cases} \overline{w}_n = \overline{v}_n - g_0^{-1}(\overline{v}_n)g(\overline{v}_n), \\ \overline{b}_n = b(n, g, g_0) = g_0^{-1}(\overline{v}_n)(g'(\overline{v}_n + \frac{2}{3}(\overline{w}_n - \overline{v}_n)) - g'(\overline{v}_n)), \\ \overline{v}_{n+1} = \overline{w}_n - \frac{3}{4}\overline{b}_n(1 - \frac{3}{2}\overline{b}_n)(\overline{w}_n - \overline{v}_n) \end{cases} \tag{2.28}$$

A simple induction argument shows that, if $M_0 < K$, then

$$\overline{v}_n < v_n, \tag{2.29}$$

$$\overline{w}_n < w_n, \tag{2.30}$$

$$\overline{w}_n - \overline{v}_n < w_n - v_n, \tag{2.31}$$

$$\overline{v}_{n+1} - \overline{w}_n < v_{n+1} - w_n \tag{2.32}$$

and

$$\overline{v}^* \leq v^*, \tag{2.33}$$

where

$$\overline{v}^* = \lim_{n \longrightarrow \infty} \overline{v}_n.$$

Note also that if $M_0 = K$, then $\overline{v}_n = v_n, \overline{w}_n = w_n$.

Case 2. In view of the upper bound for $\|F(x_{n+1})\|$ obtained in Theorem 2.1 in [4] and (2.25), $\{t_n\}, \{s_n\}$ given in (3.9) and (3.10) are also even more precise majorizing sequences for $\{x_n\}$ and $\{y_n\}$. Therefore, if they converge under certain conditions (see Lemma 3.2), then we can produce a new semilocal convergence theorem for (JM) with sufficient con- vergence conditions or bounds that can be better than the ones of Theorem 2.4 (see also Theorem 3.4 and Example 3.5). Similar favorable comparisons (due to (2.24)) can be made with other iterative methods of the fourth order [1, 11].

CHAPTER III

EXISTING METHODS

There is no ambiguity that the quadratically convergent Newton method (NM) is one of the best root finding methods based on two evaluations of function for approximating the solution of a nonlinear equation $F(x) = 0$ and is given as

Let an iterative method be of the form

$$x_{n+1} = F(x_n),\ k = 0, 1, 2, 3, \ldots$$

here x_n is an approximation to the zero α and F is an iteration function. The iterative method starts with an initial guess x_0 and at every step we use only the last known approximate.

The classical Newton's method or one-step of Newton's method is well-known,

$$x_{n+1} = x_n - F(x_n)F^{-1}(x_n),$$

where $F^{-1}(x_n)$ denotes $\dfrac{1}{F'(x_n)}$.

The double-Newton's method or two-step of Newton's method [1] is considered,

$$\begin{aligned} y_n &= x_n - F(x_n)F^{-1}(x_n), \\ x_{n+1} &= y_n - F(y_n)F^{-1}(y_n). \end{aligned}$$

In 1969, The Jarratt method(JM) [6], which has fourth-order convergence, is given by

$$\begin{aligned} y_n &= x_n - \frac{2}{3}F(x_n)F^{-1}(x_n), \\ J_f(x_n) &= \frac{3F'(y_n)+F'(x_n)}{6F'(y_n)-2F'(x_n)} \\ x_{n+1} &= x_n - J_f(x_n)F(x_n)F^{-1}(x_n). \end{aligned}$$

In 2008, Wang et al. [14] improved the Jarratt method as follows:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}F(x_n)F^{-1}(x_n), \\ J_f(x_n) &= \frac{3F'(y_n)+F'(x_n)}{6F'(y_n)-2F'(x_n)} \\ z_n &= x_n - J_f(x_n)F(x_n)F^{-1}(x_n) \\ x_{n+1} &= z_n - F(z_n)F^{-1}(z_n). \end{aligned}$$

In 2009, Wang et al. [14] improved the Jarratt method as follows:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}F(x_n)F^{-1}(x_n), \\ J_f(x_n) &= \frac{3F'(y_n)+F'(x_n)}{6F'(y_n)-2F'(x_n)} \\ F^{-1}(y_n) &= \frac{2F'(x_n)F'(y_n)}{3F'(x_n)-F'(y_n)} \\ x_{n+1} &= y_n - F(y_n)F^{-1}(y_n). \end{aligned}$$

The Jarratt method(JM) [1, 4], which has fourth-order local convergence analysis , is given by

$$\begin{aligned} & y_n = x_n - F(x_n)F^{-1}(x_n), \\ & z_n = F^{-1}(x_n)\Big[F'\Big(x_n + \frac{2}{3}(y_n - x_n)\Big) - F'(x_n)\Big]. \\ & x_{n+1} = y_n - \frac{3}{4}z_n\Big(I - \frac{3}{2}z_n\Big)(y_n - x_n) \end{aligned} \tag{3.34}$$

for each $n \geq 0$. The fourth order of (JM) is the same as that of a two-step Newton method [1, 4]. But the computational cost is less than that of Newton's method.

CHAPTER IV

CONVERGENCE ANALYSIS

The new method is proposed with a fifth-order convergence. To consider the iteration scheme, we have

$$\begin{aligned} y_n &= x_n - F(x_n)F^{-1}(x_n), \\ z_n &= y_n - F(y_n)F^{-1}(y_n), \\ x_{n+1} &= z_n - F(z_n)F^{-1}(z_n) \end{aligned} \tag{4.35}$$

which is the triple-Newton's method.

Theorem 4.6. *Let x^* be a simple zero of adequately differentiable function $F : I \subseteq R \rightarrow R$ for an open interval I. If x_0 is adequately close to x^*, then the new method defined by 4.35 is of fifth-order and satisfies the error equation*

$$e_{n+1} = \frac{1}{4}e_n + \frac{9}{16}e_n^2 - (7C_2C_3 - 6C_2^3 + C_4)e_n^4 + (4C_2C_3 - 4C_2^3 + 8C_2^4 - 8C_2^2C_3)e_n^5 + O(e_n^6),$$

where $e_n = x_n - x^*$ and $C_k = \frac{1}{k!}F^{(k)}(x^*)F^{-1}(x^*)$.

Proof Using Taylor expansion, we can get

$$F(x_n) = F'(x^*)[e_n + C_2e_n^2 + C_3e_n^3 + O(e_n^4)] \tag{4.36}$$

and

$$F'(x_n) = F'(x^*)[1 + 2C_2e_n + 3C_3e_n^2 + O(e_n^3)]. \tag{4.37}$$

Dividing the new two expansions (4.36) and (4.37) on each other, gives us

$$
\begin{aligned}
F(x_n)F^{-1}(x_n) &= e_n - C_2e_n^2 + 2(C_2^2 - C_3)e_n^3 \\
&\quad +(7C_2C_3 - 4C_2^3 + C_4)e_n^4 + O(e_n^5) && (4.38)\\
F(y_n) &= F'(x^*)[C_2e_n^2 - 2(C_2^2 - C_3)e_n^3 \\
&\quad +C_2^2e_n^4 - 4C_2(C_2^2 - C_3)e_n^5 + O(e_n^6)] && (4.39)\\
F'(y_n) &= F'(x^*)[1 + 2C_2^2e_n^2 - 4C_2(C_2^2 - C_3)e_n^3 + 3C_2^2C_3e_n^4 \\
&\quad -12C_2C_3(C_2^2 - C_3)e_n^5 + O(e_n^6)] && (4.40)\\
F(y_n)F^{-1}(y_n) &= C_2e_n^2 - 2(C_2^2 - C_3)e_n^3 + (C_2^2 - 2C_2^3)e_n^4 + (8C_2^2 - 4C_2)(C_2^2 - C_3)e_n^5 \\
&\quad +O(e_n^6) && (4.41)\\
F(z_n) &= F'(x^*)[-(7C_2C_3 - 6C_2^3 + C_4)e_n^4 \\
&\quad +(4C_2C_3 - 4C_2^3 + 8C_2^4 - 8C_2^2C_3)e_n^5 + O(e_n^6)] && (4.42)\\
F'(z_n) &= F'(x^*)[-4(7C_2C_3 - 6C_2^3 + C_4)e_n^3 \\
&\quad +5(4C_2C_3 - 4C_2^3 + 8C_2^4 - 8C_2^2C_3)e_n^4 + O(e_n^5)] && (4.43)\\
F(z_n)F^{-1}(z_n) &= -\frac{1}{4}e_n - \frac{9}{16}e_n^2 + O(e_n^3). && (4.44)
\end{aligned}
$$

From (4.38−4.44), we obtain

$$
\begin{aligned}
e_{n+1} &= \frac{1}{4}e_n + \frac{9}{16}e_n^2 - (7C_2C_3 - 6C_2^3 + C_4)e_n^4 \\
&\quad +(4C_2C_3 - 4C_2^3 + 8C_2^4 - 8C_2^2C_3)e_n^5 + O(e_n^6) && (4.45)
\end{aligned}
$$

This means the method defined by (4.35) are of fifth-order. That completes the proof.

CHAPTER V

SEMILOCAL CONVERGENCE ANALYSIS

The semilocal convergence of Jarratt method using recurrent functions.

Lemma 5.7. *Under the ($\boldsymbol{C}$) conditions, choose*

$$\phi \in [\phi_0, \phi_1] \text{ if } \phi_1 \neq 1 \quad \text{and} \quad \phi \in [\phi_0, 1) \text{ if } \phi_1 = 1. \tag{5.46}$$

Then the scalar sequences $\{s_n\}, \{t_n\}$ *given by*

$$\begin{aligned} t_0 &= 0, \qquad s_0 = \eta \\ t_{n+1} &= s_n + \frac{M(1 + M(s_n - t_n))(s_n - t_n)^2}{2(1 - M_0 t_n)^2} \\ s_{n+1} &= t_{n+1} + \frac{1}{1 - M_0 t_{n+1}} \left[\frac{M(t_{n+1} - s_n)^2}{2} + \frac{13L(s_n - t_n)^4}{108} \right. \\ &\quad \left. + \frac{NM(s_n - t_n)^4}{9(1 - M_0 t_n)} + \frac{M^3(s_n - t_n)^4}{3(1 - M_0 t_n)^2} \right] \end{aligned} \tag{5.47}$$

are non-decreasing, bounded from above by

$$t^{**} = \left(1 + \frac{\alpha}{1 - \phi}\right)\eta \tag{5.48}$$

and converge to their unique least upper bound $t^* \in [0, t^{**}]$. *Moreover, the following estimate holds:*

$$0 \leq s_{n+1} - t_{n+1} \leq \phi(s_n - t_n), \tag{5.49}$$

where

$$\alpha = \frac{M(1 + M\eta)\eta}{2}$$

.

Proof. We show, using induction on k, that

$$0 \leq \frac{M(1 + M(s_k - t_k))(s_k - t_k)}{2(1 - M_0 t_k)^2} \leq \alpha \tag{5.50}$$

and

$$0 \leq \frac{1}{1-M_0t_{n+1}}\left[\frac{M\alpha^2}{2}(s_k-t_k)+\frac{13L}{108}(s_k-t_k)^3+\frac{NM(s_k-t_k)^3}{9(1-M_0t_k)}+\frac{M^3(s_k-t_k)^3}{3(1-M_0t_k)^2}\right]$$
$$\leq \phi. \tag{5.51}$$

The estimate (5.50) holds for $k=0$ by the choice of α. Moreover, the estimates (5.49) and (5.51) hold for $n=0$ by (5.47), the choice of ϕ_0 and (5.7). Let us assume (5.49) – (5.51) hold for all $k \leq n$. We have in turn by the induction hypotheses:

$$\begin{aligned} s_k-t_k &\leq \phi(s_{k-1}-tk-1)\leq\cdots\leq\phi(s_0-t_0)=\phi^k\eta, \\ t_{k+1} &\leq s_k+\alpha(s_k-t_k) \\ &\leq t_k+\alpha\phi^k\eta+\phi^k\eta \\ &\leq s_{k-1}+\alpha(s_{k-1}-t_{k-1})+\alpha\phi^k\eta+\phi^k\eta \\ &\leq s_{k-1}+\alpha\phi^{k-1}\eta+\alpha\phi^k\eta+\phi^k\eta \\ &\leq \cdots \\ &\leq s_0+\alpha\eta(1+\cdots+\phi^k)+\phi^k\eta \\ &< \eta+\frac{\alpha\eta(1-\phi^{k+1})}{1-\phi}+\phi^k\eta, \end{aligned}$$

$$\frac{M(s_k-t_k)}{2(1-M_0t_k)^2}+\frac{M^2(s_k-t_k)^2}{2(1-M_0t_k)^2}\leq\alpha$$

or

$$\frac{2}{M^2}\frac{M}{2}\frac{(s_k-t_k)}{(1-M_0t_k)^2}+\frac{(s_k-t_k)^2}{(1-M_0t_k)^2}\leq\frac{2\alpha}{M^2},$$

or

$$\left(\frac{s_k-t_k}{1-M_0t_k}\right)^2\leq\frac{2\alpha}{M^2},$$

and

$$\begin{aligned}
\frac{NM(s_k - t_k)^3}{9(1 - M_0 t_k)} &= \frac{NM(1 - M_0 t_k)(s_k - t_k)^3}{9(1 - M_0 t_k)^2} \\
&= \frac{NM}{9}(1 - M_0 t_k)\left(\frac{s_k - t_k}{1 - M_0 t_k}\right)^2 (s_k - t_k) \\
&\leq \frac{2}{M^2}\frac{NM}{9}(s_k - t_k)\alpha = \frac{2N\alpha}{9M}(s_k - t_k), \\
\frac{M^3(s_k - t_k)^3}{3(1 - M_0 t_k)^2} &= \frac{M^3(s_k - t_k)}{3}\frac{(s_k - t_k)^3}{(1 - M_0 t_k)^2} \\
&\leq \frac{M^3}{3}(s_k - t_k)\frac{2\alpha}{M^2} = \frac{2M\alpha}{3}(s_k - t_k).
\end{aligned}$$

Hence, instead of (5.51), we can show

$$\begin{aligned}
0 &\leq \frac{1}{1 - M_0 t_{k+1}}\left[\frac{M\alpha^2}{2}(s_k - t_k) + \frac{13L}{108}(s_k - t_k)^3 + \frac{2N\alpha}{9M}(s_k - t_k) + \frac{2M\alpha}{3}(s_k - t_k)\right] \\
&\leq \phi. \qquad (5.52)
\end{aligned}$$

The estimate (5.50) can be written as

$$M(1 + M\phi^k\eta)\phi^k\eta \leq 2\alpha(1 - M_0 t_k)^2$$

or

$$M(1 + M\phi^k\eta)\phi^k\eta \leq 2\alpha + 2\alpha M_0^2 t_k^2 - 4M_0\alpha t_k.$$

So, we can show, instead of (5.50),

$$M(1 + M\phi^k\eta)\phi^k\eta + 4M_0\alpha t_k \leq 2\alpha$$

or

$$M(1 + M\phi^k\eta)\phi^k\eta + 4M_0\alpha\left[\eta + \alpha\eta\left(\frac{1 - \phi^k}{1 - \phi}\right) + \phi^{k-1}\eta\right] - 2\alpha \leq 0. \qquad (5.53)$$

The estimate (5.53) motivates us to define polynomials $\bar{f}_k$ on $[0, 1)$ (for $\phi = t$) by

$$\begin{aligned}
\bar{f}_k(t) &= M(1 + M\phi^k\eta)t^k\eta + 4M_0\alpha\eta\left[1 + \alpha\left(\frac{1 - t^k}{1 - t}\right) + t^{k-1}\right] - 2\alpha \\
&= Mt^k\eta + M^2\eta^2 t^{2k} + 4M_0\alpha\left[1 + \alpha\left(\frac{1 - t^k}{1 - t}\right) + t^{k-1}\right]\eta - 2\alpha \qquad (5.54)
\end{aligned}$$

or, since $t^2 \leq t$ for $t \in [0, 1]$, define the polynomials f_k on $[0, 1)$ by

$$f_k(t) = Mt^k\eta + M^2\eta^2 t^k + 4M_0\alpha\left[1 + \alpha\left(\frac{1 - t^k}{1 - t}\right) + t^{k-1}\right]\eta - 2\alpha \qquad (5.55)$$

We need a relationship between two consecutive polynomials f_k :

$$\begin{aligned} f_{k+1}(t) &= Mt^{k+1}\eta + M^2\eta^2 t^{k+1} + 4M_0\alpha\left[1+\alpha\left(\frac{1-t^{k+1}}{1-t}\right)+t^k\right]\eta - 2\alpha + f_k(t) - f_k(t) \\ &= f_k(t) + g(t)t^{k-1}\eta, \end{aligned} \tag{5.56}$$

where g and its unique positive root $\phi_g \in [0,1)$ are given in Definition 2.3. The estimate (5.53) is true if

$$f_k(\phi) \leq 0 \tag{5.57}$$

or if

$$f_1(\phi) \leq 0, \tag{5.58}$$

since by (5.56) we have

$$f_k(\phi) = f_1(\phi) \tag{5.59}$$

But (5.58) is true by the definition of ϕ_{f_1} and (5.7).Define

$$f_\infty(\phi) = \lim_{k\longrightarrow\infty} f_k(\phi) \tag{5.60}$$

Then we also have

$$f_\infty(\phi) = \lim_{k\longrightarrow\infty} f_k(\phi) = \lim_{k\longrightarrow\infty} f_1(\phi) \leq \lim_{k\longrightarrow\infty} 0 = 0. \tag{5.61}$$

This completes the induction for (5.50). The estimate (5.52) is true if

$$\frac{M\alpha^2\phi^k\eta}{2} + \frac{13L}{108}(\phi^k\eta)^3 + \frac{2N\alpha}{9M}\phi^k\eta + \frac{2\alpha M}{3}\phi^k\eta \leq \phi(1-M_0t_{k+1})$$

or

$$\begin{aligned} &\frac{M\alpha^2\phi^k\eta}{2} + \frac{13L}{108}(\phi^k\eta)^3 + \frac{2N\alpha}{9M}\phi^k\eta + \frac{2\alpha M}{3}\phi^k \\ &+ \phi M_0\left[1+\alpha\left(\frac{1-\phi^{k+1}}{1-\phi}\right)+\phi^k\right]\eta - \phi \leq 0. \end{aligned} \tag{5.62}$$

The estimate (5.62) motivates us to define polynomials h_k on $[0,1)$ by

$$\begin{aligned} h_k(t) = &\frac{M\alpha^2}{2}t^k\eta + \frac{13L}{108}\eta^3 t^k + \frac{2N\alpha}{9M}t^k\eta + \frac{2\alpha M}{3}t^k\eta \\ &+ \phi M_0\left[1+\alpha\left(\frac{1-t^{k+1}}{1-t}\right)+t^k\right]\eta - \phi. \end{aligned} \tag{5.63}$$

We need a relationship between two consecutive polynomials h_k :

$$\begin{aligned} h_{k+1}(t) &= \frac{M\alpha^2}{2}t^{k+1}\eta + \frac{13L}{108}\eta^3 t^{k+1} + \frac{2N\alpha}{9M}t^{k+1}\eta + \frac{2\alpha M}{3}t^{k+1}\eta \\ &+ \phi M_0\left[1+\alpha\left(\frac{1-t^{k+2}}{1-t}\right)+t^{k+1}\right]\eta - \phi - \frac{M\alpha^2}{2}t^k\eta - \frac{13L}{108}\eta^3 t^k - \frac{2\alpha M}{3}t^k\eta \\ &- \frac{2N\alpha}{9M}t^k\eta - \phi M_0\left[1+\alpha\left(\frac{1-t^{k+1}}{1-t}\right)+t^k\right] + \phi + h_k(t) \end{aligned}$$

and so

$$f_{k+1}(t) = h_k(t) + g_1(t)t^k\eta, \tag{5.64}$$

where g_1 and the unique positive root ϕ_{g_1} are given in Definition 2.3. The estimate (5.62) is true if

$$h_k(\phi) \leq 0 \tag{5.65}$$

or, if

$$h_1(\phi) \leq 0 \tag{5.66}$$

since

$$h_k(\phi) = h_1(\phi). \tag{5.67}$$

But (5.66) is true by the definition of ϕ_{h_1} and (5.7). Define a function h_∞ on $[0,1)$ by

$$h_\infty(\phi) = \lim_{k\longrightarrow\infty} h_k(\phi). \tag{5.68}$$

Then we have

$$h_\infty(\phi) = \lim_{k\longrightarrow\infty} h_k(\phi) = \lim_{k\longrightarrow\infty} h_1(\phi) \leq \lim_{k\longrightarrow\infty} 0 = 0. \tag{5.69}$$

This completes the induction for (5.7) – (5.51). It follows that the sequences $\{s_n\}$ and $\{t_n\}$ are non-decreasing, bounded from above by t^{**} given in a closed form by (5.48) and converge to their unique least upper bound $t^* \in [0, t^{**}]$. This completes the proof. □

Lemma 5.8. *[1, 4] Under the hypotheses of Lemma 5.7, further assume*

$$\sqrt[3]{b}\eta \leq 1, \tag{5.70}$$

where

$$b = a + \frac{M^3}{8} \quad \text{and} \quad a = \frac{M^3}{8} + \frac{NM}{q} + \frac{13L}{108}. \tag{5.71}$$

Fix

$$q \in \left(\sqrt[3]{b}, \frac{1}{\eta}\right), \quad \eta \neq 0. \tag{5.72}$$

Define the parameters p_0, p by

$$\left.\begin{aligned} p_0 &= \frac{1}{M_0}\left(1 - \frac{\sqrt[3]{b}}{q}\right), \\ p &= \frac{Mq}{2\sqrt[3]{b}}, \end{aligned}\right\} \tag{5.73}$$

and a function g_3 on $[1, \frac{1}{q})$ by

$$g_3(t) = t + \frac{1}{q} + \frac{p}{q^2}\left(\frac{(qt)^2}{1-(qt)^2} + t^2\right). \tag{5.74}$$

Moreover, assume

$$\min\{t_1, g_3(\eta)\} \leq p_0. \tag{5.75}$$

Then the following estimates hold for all $k0$:

$$\left.\begin{aligned} t_{k+1} - s_k &\leq \frac{p}{q^2}\sqrt{(q\eta)^{4^{k+1}}}, \\ s_k - t_k &\leq \frac{1}{q}(q\eta)^{4^k}. \end{aligned}\right\} \tag{5.76}$$

Proof. We show

$$s_{m+1} - t_{m+1} \leq q^3(s_m - t_m)^4. \tag{5.77}$$

If the estimate (5.77) holds, then we have

$$\begin{aligned} q(s_{m+1} - t_{m+1}) &\leq (q(s_m - t_m))^{4^1} \\ &\leq (q\eta)^{4^{m+1}}, \end{aligned} \tag{5.78}$$

which implies the second equation in (5.76). We have the estimate

$$
\begin{aligned}
&\frac{M}{2}(t_{m+1}-s_m)^2+\frac{13L}{108}(s_m-t_m)^4+\frac{NM}{9(1-M_0t_m)}(s_m-t_m)^4+\frac{M^3}{3(1-M_0t_m)^2}(s_m-t_m)^4\\
&\le \frac{M}{2}\left(\frac{M(s_m-t_m)^2}{2(1-M_0t_m)}\right)^2+\frac{13L}{108}(s_m-t_m)^4\\
&\qquad+\frac{NM}{9(1-M_0t_m)}(s_m-t_m)^4+\frac{M^3}{3(1-M_0t_m)^2}(s_m-t_m)^4\\
&\le \frac{M^3}{8}\frac{(s_m-t_m)^4}{(1-M_0t_m)^2}+\frac{13L}{108}\frac{(s_m-t_m)^4}{(1-M_0t_m)^2}(1-M_0t_m)^2\\
&\qquad+\frac{NM(s_m-t_m)^4}{9(1-M_0t_m)^2}(1-M_0t_m)+\frac{M^3}{3}\frac{(s_m-t_m)^4}{(1-M_0t_m)^2}\\
&\le \frac{b(s_m-t_m)^4}{(1-M_0t_m)^2},
\end{aligned}
$$

that is, we have

$$s_{m+1}-t_{m+1}\le\frac{b(s_m-t_m)^4}{(1-M_0t_{m+1})(1-M_0t_m)}.$$

Instead of showing (5.77), we can show

$$\frac{b(s_m-t_m)^4}{(1-M_0t_m)^2(1-M_0t_{m+1})}\le q^3(s_m-t_m)^4 \tag{5.79}$$

or

$$\frac{b}{(1-M_0t_{m+1})^3}\le q^3, \tag{5.80}$$

or

$$t_{m+1}\le p_0. \tag{5.81}$$

By the hypothesis (5.75), we have

$$t_1\le p_0. \tag{5.82}$$

Assume

$$t_m\le p_0. \tag{5.83}$$

We also have

$$
\begin{aligned}
t_{m+1}-s_m &= \frac{M(s_m-t_m)^2}{2(1-M_0t_m)}\\
&\le \frac{Mq}{2\sqrt[3]{b}}(s_m-t_m)^2=p(s_m-t_m)^2.
\end{aligned} \tag{5.84}
$$

We get in turn

$$\begin{aligned}
t_{m+1} &\leq (s_m - t_m) + (t_m - s_{m-1}) + \cdots + (t_1 - s_0) + s_0 + p(s_m - t_m)^2 \\
&\leq \eta + \frac{1}{q}(q\eta)^{4^m} + p((s_m - t_m)^2 + (s_{m-1} - t_{m-1})^2 + \cdots + (s_0 - t_0)^2) \\
&\leq \eta + \frac{1}{q}(q\eta)^{4^m} + \frac{p}{q^2}((q\eta)^{(4^m)^2} + (q\eta)^{(4^{m-1})^2} + \cdots + \eta^2) \\
&= \eta + \frac{1}{q}(q\eta)^{4^m} + \frac{p}{q^2}(((q\eta)^{\frac{1}{2}})^{4^{m+1}} + ((q\eta)^{\frac{1}{2}})^{4^m} + \cdots + ((q\eta)^{\frac{1}{2}})^{4^1} + \eta^2) \\
&\leq \eta + \frac{1}{q} + \frac{p}{q^2}((q\eta)^{2(m+1)} + (q\eta)^{2m} + \cdots + (q\eta)^2 + \eta^2) \\
&\leq \eta + \frac{1}{q}\left(\frac{(q\eta)^2}{1 - (q\eta)^2} + \eta^2\right) = g_3(\eta) \leq p_0, \qquad (5.85)
\end{aligned}$$

which completes the induction for (5.81). This completes the proof. □

Theorem 5.9. *Under the hypotheses (2.2)*–*(5.47) and (5.65), further assume that the hypotheses of Lemma 5.7 hold and*

$$\overline{U}(x, t^*) \subseteq D \qquad (5.86)$$

Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (JM) are well defined, remain in $\overline{U}(x, t^*)$ for all $n \leq 0$ and converge to a unique solution x^* of the equation $F(x) = 0$ in $\overline{U}(x, t^*)$. Moreover, the following estimates hold:

$$\begin{aligned}
\|y_n - x_n\| &\leq s_n - t_n, \\
\|x_{n+1} - y_n\| &\leq t_{n+1} - s_n, \\
\|x_n - x^*\| &\leq t^* - t_n, \\
\|y_n - x^*\| &\leq t^* - s_n,
\end{aligned}$$

Furthermore, under the hypotheses of Proposition 5.8 , the estimates (5.76) also hold. Finally, if $R \geq t^*$ such that

$$U(x, R) \subseteq \mathcal{D}$$

and

$$R \leq \frac{2}{M_0} - t^*,$$

then the solution x^* *is unique in* $U(x, R)$.

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   New Convergence Theorems for Equilibrium and Fixed Point Problems
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   - Fixed point theory

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(1) Nonlinear Iteration methods for addressing ofnonlinear mappings with application to optimization	ศูนย์ความเป็นเลิศทางด้านคณิตศาสตร์ (23 ก.ค.2553 - 22ก.ค.2556 (3 ปี))	2,400,000
(2) The extragradient hybrid steepest descent methods for system of mixed equilibrium problems and system of variational inequality problems involving nonlinear mappings	ทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่ ปี 2554 (สกว.) (15 มิ.ย.2554 - 14 มิ.ย.2556 (2 ปี)) (หัวหน้าโครงการ)	420,000
(3) Iterative Schemes for Fixed Point Problems and System of Generalized Equilibrium Problems of Nonlinear Mappings in Hilbert Spaces	ทุนวิจัยมหาวิทยาลัยเทคโนโลยีราชมงคลรัตนโกสินทร์ (งบประมาณเงินรายได้ พ.ศ. 2554) (1 ต.ค. 2553 – 30 ก.ย. 2554 (หัวหน้าโครงการ)	30,000
(4) Approximation of Optimization Problems and System of Generalized Mixed Equilibrium Problems with Special Implicit Algorithm	ทุนวิจัยมหาวิทยาลัยเทคโนโลยีราชมงคลรัตนโกสินทร์ (งบประมาณเงินแผ่นดิน พ.ศ. 2555) (1 ต.ค. 2554 – 30 ก.ย. 2555 (หัวหน้าโครงการ)	100,000

7. การเผยแพร่ผลงานวิจัย

7.1 ผลงานวิจัยที่ตีพิมพ์ในวารสารวิชาการนานาชาติ

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7.2 ผลงานวิชาการอื่น ๆ (เช่น Proceedings หนังสือ ฯลฯ)

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7.3 Domestic Conference (Oral Presentations)

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(2) ชัยชนะ ใจบุญ, ภูมิ คำเอม และอุษา ฮัมฟรี่, 2551, "An Extragradient Method for Solving Equilibrium and Fixed Point Problems in Hilbert Spaces", การประชุมวิชาการทางคณิตศาสตร์ ประจำปี 2551 (ครั้งที่ 13), 6-7 พฤษภาคม, มหาวิทยาลัยศรีนครินทรวิโรฒ, กรุงเทพฯ.

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(5) ชัยชนะ ใจบุญ, 2553, "Modified viscosity hybrid steepest descent methods for systems of mixed equilibrium problems and semigroups of nonexpansive mappings with application to

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7.4 รางวัลที่เคยได้รับ (ด้านวิชาการโดยเฉพาะอย่างยิ่งที่เกี่ยวกับงานวิจัย)

- รางวัลการนำเสนอผลงานโปสเตอร์ดีเด่น กลุ่ม 4 สาขาคณิตศาสตร์ สถิติ และคณิตศาสตร์ศึกษา ชัยชนะ ใจบุญ และ ภูมิ คำเอม, 2552, วิธีการประมาณค่าเอ็กตร้าเกรเดียนต์แบบหน่วงสำหรับปัญหาอสมการเชิงการแปรผันแบบริแลคซโคโดเอ๊อซิฟ ปัญหาดุลยภาพและปัญหาจุดตรึงของวงนับได้ของการส่งแบบไม่ขยาย Poster P-M17, การประชุมทางวิชาการ “วิทยาศาสตร์วิจัย” ครั้งที่ 2, 9-10 มีนาคม, 2552 คณะวิทยาศาสตร์, มหาวิทยาลัยนเรศวร, จ.พิษณุโลก, หน้า 203.
- รางวัลผลการศึกษายอดเยี่ยมชั้นวิทยาศาสตร์ดุษฎีบัณฑิตและมีผลงานตีพิมพ์ในวารสารวิชาการที่มี Impact Factor รวมกันแล้ว ได้แต้มสูงสุด 3 อันดับแรก
ของสาขาคณิตศาสตร์ โดยเปรียบเทียบกันทุกมหาวิทยาลัยทั่วประเทศไทย ประจำปี
2553 โดย มูลนิธิศาสตราจารย์ ดร. แถบ นีละนิธิ
- รางวัลสภาวิจัยแห่งชาติ : รางวัลวิทยานิพนธ์ ประจำ 2554 รางวัลระดับดี
สาขาวิทยาศาสตร์กายภาพและคณิตศาสตร์
เรื่อง ทฤษฎีการลู่เข้าใหม่สำหรับปัญหาดุลยภาพและปัญหาจุดตรึง